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The fundamental solution for small steady three-dimensional disturbances to a two-dimensional parallel shear flow

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1. INTRODUCTION

In addition to the enormous aerodynamical literature concerned with the disturbances to a uniform stream produced by the presence of obstacles, there is a very extensive literature on disturbances to irrotational non-uniform streams. The smaller body of work on disturbances to rotational streams is concerned principally with two-dimensional disturbances to a parallel shear flow; these are fairly easy to treat because the vortex lines are straight; they remain parallel and do not get stretched at all during the motion.

By contrast, in flows where the disturbances are three-dimensional in character (even though the undisturbed stream may still be two-dimensional), stretching and rotation of the vortex lines must play an important part. In spite of the difficulties which this introduces, certain outstanding papers on the theory of these flows have appeared.

Squire (1933) found the equation governing small three-dimensional disturbances to a two-dimensional parallel flow. If the velocity field is

$$(V(y)+u, v, w), \tag{1}$$

then by neglecting squares of the disturbance velocities u, v, w and analysing them into Fourier components, he showed that disturbances varying with x and z like $e^{i(\alpha x+\beta z)}$ must vary with time like $e^{-i\alpha ct}$, where c is such that the equation

$$(V-c)\{v''-(\alpha^2+\beta^2)v\}-V''v-(\nu/i\alpha)\{v^{\mathrm{iv}}-2(\alpha^2+\beta^2)v''+(\alpha^2+\beta^2)^2v\}=0 \quad (2)$$

(where ν is kinematic viscosity and where primes signify differentiation with respect to y) is satisfied (together with boundary conditions) by a non-vanishing value of the y-velocity v. Values of c with negative imaginary part correspond to stable disturbances, values with positive imaginary part to unstable disturbances, and real values to neutral disturbances. Equation (2) shows that neutral disturbances with $\beta=\beta_0\neq 0$, $\alpha=\alpha_0$, $\nu=\nu_0$ can exist, if and only if neutral disturbances with

$$\beta=0, \qquad \alpha=\sqrt{(\alpha_0^2+\beta_0^2)}, \qquad \nu=\nu_0\sqrt{(1+\beta_0^2/\alpha_0^2)}$$

exist; the latter are two-dimensional disturbances at a lower Reynolds number*. Squire deduced the classical theorem that two-dimensional disturbances become unstable at a lower Reynolds number than three-dimensional disturbances.

Kármán & Tsien (1945) investigated small disturbances to a three-dimensional parallel stream (so that V in (1) is a function of z as well as of y) but limited themselves (as we shall do in this paper) to steady disturbances with viscosity neglected, evolving the theory of the 'lifting line' in such a non-uniform stream. They found it convenient to work from the equation for the pressure p, namely

$$\operatorname{div}\left(\frac{\operatorname{grad} p}{V^2}\right)=0, \qquad (3)$$

both because it in this case is simpler than that for any of the velocity components, and because the boundary condition in the lifting-line problem is expressed most conveniently in terms of p. The results of the lifting-line theory are complicated, however, and only in the case when V is a function of y alone (the spanwise coordinate) do they become reasonably tractable.

Another approach to the study of steady three-dimensional disturbances to a parallel shear flow has been to regard them as a 'secondary flow', to be analysed by the method initiated by Squire & Winter (1951). In this method there is no assumption that the disturbances are small; for example, the disturbance due to a sphere can be treated (Hawthorne & Martin 1955; Lighthill 1956). There is, however, an assumption that the undisturbed stream is weakly sheared. The 'primary flow' (or first approximation to the real flow) is taken to coincide (either wholly, or at least as regards its streamline pattern) with that in which the undisturbed stream is uniform. The secondary flow is a perturbation on this, due to allowing a small shear in the undisturbed stream. This shear is usually taken to be uniform, although such an assumption is not essential to the method; we shall call secondary flows calculated on this assumption 'simple-shear secondary flows'. The secondary vorticity can be calculated either direct from

* Equation (2) with $\beta=0$ is the Orr–Sommerfeld equation.

Helmholtz's equations (Hawthorne 1951, 1954), or by geometrical considerations (Preston 1954; Lighthill 1956) from the effect of the primary flow in stretching and rotating the vortex lines of the undisturbed shear flow. (From this point of view, the approximation adopted consists in neglecting the fact that it is really the exact flow, not the approximate 'primary flow', which carries the vortex lines along with it.) The calculation of the secondary velocity field from the associated vorticity field is more arduous, but can be achieved in special cases (Lighthill 1956, 1957 b).

The work of this paper originated from a desire to clear up a difficulty arising in the application of the secondary-flow method to cases where the undisturbed stream is unbounded. The difficulty is that the secondary-flow disturbance due to the presence of an obstacle falls off more slowly with distance from it than does the primary-flow disturbance. For example, a 'half-body' extending from the origin to infinity downstream has a primary-flow disturbance of 'source' type, falling off like r^{-2} (where r is distance from the origin), but the corresponding secondary-flow disturbance falls off like r^{-1}. If the procedure of successive approximation were continued, and a tertiary-flow field calculated from the configuration of vortex lines as distorted by the sum of the primary and secondary flows, this would not even tend to zero as $r \to \infty$.

As another example, a finite body makes a primary-flow disturbance of 'doublet' type, falling off like r^{-3}, but the corresponding secondary-flow disturbance falls off like r^{-2} (see Lighthill (1956), where also a not quite accurate attempt was made to calculate the tertiary flow, but the conclusion that it falls off like r^{-1} was right).

Now, there is no experimental evidence (e.g. from the flow visualizations reported by Livesey (1956)) that secondary flows are large far from the obstacle, and it seems clear that, as suggested in Lighthill (1956), the approximation sequence is not uniformly valid in this region, where (therefore) the first term gives no indication of the real behaviour of the disturbance.

There is some analogy in this to the difficulties concerning flow about an obstacle at very low Reynolds numbers, often described as 'Stokes's and Whitehead's paradoxes'. In those problems a direct procedure of expansion in powers of Reynolds number gives nonsensical results, at any rate far from the obstacle (although for three-dimensional obstacles the first term in the expansion has value near the obstacle). The difficulty was resolved by Oseen, who showed that the correct form of the equations of motion at large distances is that obtained by neglecting squares of disturbances to the uniform stream but not discarding terms of lower order in the Reynolds number.

Similarly in the present problem the true behaviour of the flow far from the body is found from the full equations with the squares of the disturbances neglected. Such an approach is obviously valid sufficiently far from the obstacle, and the only problem is how the results obtained should be related to secondary-flow and other results obtained by the methods described above.

The answer found below is that the small-disturbance solution is valid in a region, far from the obstacle, which overlaps the region, near the obstacle, where the small-shear solution (more accurately, the 'simple-shear secondary flow') is valid. This overlapping holds in the sense that the asymptotic behaviour of the small-disturbance solution as one goes near the obstacle is identical with the asymptotic behaviour of the small-shear solution as one goes away from the obstacle. The overlap makes it possible to say how the flow behaves everywhere.

These are good reasons for studying small steady three-dimensional disturbances to a parallel shear flow. Obviously, however, there are many others: to investigate, for example, the flow of sheared winds over hills. Again, one could follow up the work of Kármán & Tsien (1945) by studying further kinds of perturbation of parallel shear flows by thin shapes of aerodynamic interest.

As a basic tool for any such investigation, we seek in this paper the 'fundamental solution' of the equation. This represents the flow due to a source (but a weak source, since our equation treats only small disturbances) in a parallel shear flow. This solution will give as it stands the asymptotic behaviour of the disturbances due to a 'half-body', since these are equivalent far from the body to those produced by a source. Also, a simple differentiation of the solution will give the disturbance due to a doublet, which will represent the asymptotic behaviour of the disturbances due to a finite body. Finally, more complicated solutions of the equations can be built up (on familiar lines) once the fundamental solutions for a source and a doublet are known; work on these lines, however, is excluded from the present paper.

We limit ourselves (like Squire 1933) to the case when the undisturbed flow is two-dimensional, its velocity field being $(V(y), 0, 0)$. The formal theory for the undisturbed velocity field $(V(y, z), 0, 0)$ (used by Kármán & Tsien 1945) can be derived, but so much extra effort is then needed to deduce any concrete results (as indeed they also found) that it is better to try to keep the theory intelligible by confining discussion to the simpler case. In this case the equation, after Fourier transforms with respect to x and z have been taken, becomes Squire's equation (2) with $c = 0$ (because the disturbances are steady). There are, of course, no continuous solutions of this equation which satisfy the appropriate boundary conditions (such solutions exist only for one value of c, which in practice is non-zero); on the other hand, to represent a source (say, at the origin), one needs a solution with a discontinuity of a certain specified kind at $y = 0$. This solution is the Fourier transform of the required disturbance field, which must be deduced from it by Fourier's inversion theorem.

The Orr–Sommerfeld equation has in the past been used almost exclusively for studying harmonic disturbances. The building up of anharmonic disturbance fields by Fourier synthesis from solutions of that equation, as in this paper and the author's previous paper on boundary layers and upstream influence in supersonic flow (Lighthill 1953), is an approach that seems fruitful and may have a number of applications.

In the present paper the viscous terms in the equation are neglected. This is known to be permissible except where $V(y)$ is near the 'critical' values 0 and c, which coincide in the case here studied. Thus, shear layers in which the undisturbed velocity falls to zero anywhere are excluded in this paper as a result of the neglect of the viscous terms; such shear layers include boundary layers and jets. Boundary layers could be treated, provided that the viscous terms in the equation were retained in some 'inner viscous layer' near the wall, as used by Lighthill (1953) to provide a corrected form of boundary condition for the non-viscous equation. However, in a preliminary survey this refinement was thought to be not worth making, particularly since laminar boundary layers when disturbed are so prone to large disturbances associated with separation. With jets, again, the assumption of parallel or even nearly parallel flow at the edge, where air is being entrained, is wrong. Accordingly, the theory as set out here is limited to a wake, or a mixing region between parallel streams, or any other layer in which $V(y)$ is nowhere zero.

Since steady inviscid flow is being considered the Kármán–Tsien equation (3) could have been used. However, equation (2) for v (with $\nu = c = 0$), although no simpler than equation (3) for p, has advantages in the present problem, in which v (but not p) turns out to be a function of y and of $s = \sqrt{(x^2+z^2)}$ alone, and in which the velocity components, rather than the pressure, are the main things one wants to know.

Of the results of the theory, four main classes may be mentioned in advance. First, if the solution is expanded in ascending powers of r (distance from the source), the first term (of order r^{-2}) is the primary flow (the source flow itself) and the second (of order r^{-1}) is the small-disturbance approximation to the simple-shear secondary flow (as already mentioned) and depends only on $V(0)$ and $V'(0)$, the velocity and shear at the origin. However, the next term (which might be called the small-disturbance form of the 'tertiary flow') depends on the velocity distribution throughout the shear layer.

Another result of general interest is concerned with the behaviour of the solution for large r: a source in a shear layer produces in any region of uniform flow outside the shear layer a disturbance equivalent to a source of different strength in a different position. The strength of the equivalent source can alternatively be predicted by simpler arguments (due to Mr M. B. Glauert), in which the shear layer is regarded as a discontinuity between regions of uniform flow. The displacement in effective position is of the order of the layer width, and has not yet been predicted by such simpler arguments.

A third result is that in shear layers with small velocity spread (that is, a velocity minimum only a few per cent less than the velocity maximum) a specially simple form of solution is available; this can be given an elegant interpretation by means of images, again due to Mr M. B. Glauert. It can also be shown to be identical with the asymptotic form of the 'exact-profile secondary flow', that is, of the secondary flow calculated by allowing the

vortex lines of the exact undisturbed-flow profile $V(y)$ (not the corresponding simple-shear profile $V(0)+V'(0)y$) to be stretched and rotated by the primary flow.

Lastly, the theory is applied to the problem of the displacement of the dividing streamline in shear flows about axisymmetrical obstacles—a problem which the author has used in a number of recent papers, both as a convenient excuse for developing techniques for calculating shear flows, and as a point where theory could easily be referred to the results of experiments. In the last section of this paper it is shown what corrections need to be made, to values of this displacement as calculated from simple-shear secondary-flow theory, to take into account the improved picture of what the disturbances are like far from the obstacle.

2. Equations of motion

The momentum equation for steady incompressible inviscid flow is

$$\mathbf{v}.\nabla\mathbf{v}+\frac{1}{\rho}\nabla p=0, \tag{4}$$

where $\mathbf{v}$ is the velocity, ρ the density and p the pressure. If the departure from two-dimensional parallel flow is small, so that

$$\mathbf{v}=(V(y)+u, v, w), \tag{5}$$

where squares and products of u, v, w and their derivatives are to be neglected, equation (4) becomes

$$V(y)\frac{\partial u}{\partial x}+vV'(y)+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{6}$$

$$V(y)\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y}=0, \tag{7}$$

$$V(y)\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z}=0. \tag{8}$$

We shall assume that all disturbances vanish far upstream (at $x=-\infty$), so that (7) and (8), with the pressure eliminated, give

$$\frac{\partial}{\partial y}\{V(y)w\}=-\frac{1}{\rho}\int_{-\infty}^{x}\frac{\partial^2 p}{\partial y\partial z}\,dx=V(y)\frac{\partial v}{\partial z}. \tag{9}$$

Elimination of p from (6) and (7) gives, similarly,

$$\frac{\partial}{\partial y}\left\{V(y)\frac{\partial u}{\partial x}\right\}=V(y)\frac{\partial^2 v}{\partial x^2}-\frac{\partial}{\partial y}\{V'(y)v\}. \tag{10}$$

These equations are to be satisfied together with the equation of continuity. Now, to obtain the 'fundamental solution', due to a source of strength m at the origin, from which other solutions can be built up, we write that equation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}=m\delta(x)\delta(y)\delta(z), \tag{11}$$

where $\delta(x)$ is the delta function of Dirac, or 'impulse function'; thus, the right-hand side of (11) vanishes except at the origin, but its integral over any volume including the origin is m. No similar term appears on the right-hand side of (6), (7) or (8) because the source is taken to be a source of mass but not of momentum.

To obtain an equation for the single variable v, we first multiply (11) by $V(y)$, giving

$$V(y)\frac{\partial u}{\partial x} + V(y)\frac{\partial v}{\partial y} + V(y)\frac{\partial w}{\partial z} = mV(0)\delta(x)\delta(y)\delta(z) \tag{12}$$

(where the general theorem $f(y)\delta(y) = f(0)\delta(y)$ has been used), and then we differentiate (12) with respect to y, using (9) and (10), to obtain

$$V(y)\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial y}\{V'(y)v\} + \frac{\partial}{\partial y}\left\{V(y)\frac{\partial v}{\partial y}\right\} + V(y)\frac{\partial^2 v}{\partial z^2} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{13}$$

or

$$V(y)\nabla^2 v - V''(y)v = mV(0)\delta(x)\delta'(y)\delta(z). \tag{14}$$

Equation (14) for v is the basic equation of motion. After it has been solved, w and u may easily be deduced, from (9) in the form

$$V(y)w = \int_{\pm\infty}^{y} V(y)\frac{\partial v}{\partial z}dy, \tag{15}$$

and from the equation of continuity (11). The choice of lower limit in (15) arises from the condition that the disturbance vanish at $y = \pm\infty$, but either limit gives the same answer. This is because

$$\int_{-\infty}^{\infty} V(y)v\,dy = 0. \tag{16}$$

To prove (16) (which is a useful check on the accuracy of solutions, and has been applied as such to all those given in this paper), integrate (13) from $y = -\infty$ to $y = +\infty$. This shows that the left-hand side of (16) is a harmonic function of the two variables x and z for all values of those variables; but it vanishes at infinity, and so must be zero everywhere.

To obtain u, on the other hand, one must integrate the equation of continuity with respect to x from $-\infty$ (not $+\infty$) to x, since it is possible that fluid at $x = +\infty$, whose vorticity may have been permanently redistributed when it passed the source, may have non-zero disturbance velocity.

It is possible that the solution of the basic equation (14), for given $V(y)$, could be obtained by some sort of 'analogue method'. Thus, the equation can be written

$$\mathrm{div}\left\{V^2(y)\mathrm{grad}\frac{v}{V(y)}\right\} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{17}$$

which shows that $v/V(y)$ is proportional to the electrostatic potential of a dipole at the origin, with axis the y-axis, in a stratified medium with dielectric constant proportional to $V^2(y)$. Many similar analogies are, of course, possible, and some might, perhaps, be useful in giving intuitive information

about the solution, or an experimental procedure for computing it. The author has not so far been able to make such use of them, and in the meantime offers an analytical method of solution.

It may also be noted that the theory can be at once extended to problems with density stratification, in which the undisturbed flow has its density $\rho(y)$ as well as its velocity $(V(y), 0, 0)$ dependent on y; however, this extension is straightforward only if gravity is still negligible—thus, the variable inertia of the fluid, but not the variable weight, can easily be taken into account. The equation which results in this case, corresponding to (17), is

$$\operatorname{div}\left\{V^2(y)\frac{\rho(y)}{\rho(0)}\operatorname{grad}\frac{v}{V(y)}\right\} = mV(0)\delta(x)\delta'(y)\delta(z), \tag{18}$$

so that the expression for $v/V(y)$ may be derived from its expression for constant ρ by simply replacing $V(y)$ in the latter expression by

$$V(y)\sqrt{\{\rho(y)/\rho(0)\}}.$$

Unfortunately this result is not of much value because in problems with density stratification the variable buoyancy of the fluid is usually important (or else the Mach number is not negligible). One may hope that the theory can later be extended to take the gravitational term into account, but this would require a new investigation.

To solve equation (14) it is usually convenient to divide it by $V(y)$, giving

$$\nabla^2 v - \frac{V''(y)}{V(y)}v = m\delta(x)\delta(z)\left\{\delta'(y) + \frac{V'(0)}{V(0)}\delta(y)\right\}, \tag{19}$$

where the form of the right-hand side follows from the general theorem

$$f(y)\delta'(y) = f(0)\delta'(y) - f'(0)\delta(y).$$

One next writes v as a Fourier integral

$$v(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha x+\beta z)}v_0(y, \alpha, \beta)\, d\alpha d\beta. \tag{20}$$

Then, by substitution in (19),

$$v_0'' - \left(\alpha^2 + \beta^2 + \frac{V''}{V}\right)v_0 = \frac{m}{4\pi^2}\left\{\delta'(y) + \frac{V'(0)}{V(0)}\delta(y)\right\}, \tag{21}$$

where all primes signify differentiation with respect to y.

Apart from the delta functions on the right-hand side, representing the effect of the source, equation (21) is the same as Squire's equation (2), with c and ν put equal to 0. The fact that α and β appear only in the combination

$$\alpha^2 + \beta^2 = k^2 \tag{22}$$

is again important. It follows from it that v itself depends only on y and on

$$x^2 + z^2 = s^2. \tag{23}$$

For if we substitute

$$\alpha = k\cos\theta, \qquad \beta = k\sin\theta, \qquad x = s\cos\lambda, \qquad z = s\sin\lambda \tag{24}$$

in (20), then, since $d\alpha d\beta = k\,dk d\theta$, we obtain

$$v = \int_0^\infty v_0 k\,dk \int_0^{2\pi} e^{iks\cos(\theta-\lambda)}\,d\theta$$

$$= 2\pi \int_0^\infty v_0 k J_0(ks)\,dk. \tag{25}$$

Thus v, a function of y and s alone, is the 'Hankel transform' (25) of v_0, a function of y and k alone.

This v_0 is to be obtained from (21), which states that for $y > 0$ and $y < 0$ the simple equation

$$v_0'' - v_0\left(k^2 + \frac{V''}{V}\right) = 0 \tag{26}$$

is satisfied by v_0, but that v_0 and v_0' are discontinuous at $y = 0$, in such a way that

$$v_0(+0) - v_0(-0) = \frac{m}{4\pi^2}, \qquad v_0'(+0) - v_0'(-0) = \frac{m}{4\pi^2}\frac{V'(0)}{V(0)}. \tag{27}$$

The problem has thus been reduced to terms not involving any generalized functions.

3. Simple solutions for two particular cases

There are two particular, although somewhat artificial, distributions $V(y)$ of the undisturbed velocity for which very simple solutions exist. First, if

$$V(y) = U + Ay, \tag{28}$$

so that the oncoming flow is uniformly sheared, then (26) becomes

$$v_0'' - k^2 v_0 = 0, \tag{29}$$

and the solution of (29) which tends to zero as $|y| \to \infty$ and satisfies (27) is

$$v_0 = \frac{m}{8\pi^2} e^{-k|y|} \operatorname{sgn} y - \frac{m}{8\pi^2 k}\frac{V'(0)}{V(0)} e^{-k|y|}, \tag{30}$$

where $\operatorname{sgn} y$ is $+1$ when $y > 0$ and -1 when $y < 0$. By the result (Watson 1944, § 13.2 (1))

$$\int_0^\infty e^{-k|y|} J_0(ks)\,dk = \frac{1}{\sqrt{(y^2+s^2)}}, \tag{31}$$

and its derivative with respect to y, equations (30) and (25) give

$$v = \frac{\partial}{\partial y}\left(-\frac{m}{4\pi r}\right) - \frac{V'(0)}{V(0)}\frac{m}{4\pi r}, \tag{32}$$

where $r = \sqrt{(y^2+s^2)} = \sqrt{(x^2+y^2+z^2)}$. That the coefficient of $V'(0)/V(0)$ must be simply the potential of a source of strength m is independently obvious from equation (19) when the second term therein vanishes; similarly, the other term must be the y-derivative of the source potential $(-m/4\pi r)$.

The first term in (32) is the 'primary' flow due to the source, and the second term is the small-disturbance approximation to the simple-shear secondary flow (Lighthill 1957 a). The full secondary flow in the simple-shear case (28) is the solution obtained by assuming not small source strength but small shear; it is a perturbation on the irrotational flow due to the source, derived by neglecting the square of the shear; at a large distance from the source, where the disturbance to the oncoming stream is also small, it takes the form shown in (32). What is interesting in (32) is that no higher powers of the shear (representing tertiary, quartary flows, etc.) appear, although in the present analysis these have not been neglected. It is, as we shall see, only for the particular, and artificial, undisturbed velocity distribution (28) that this is so.

Moreover, such terms are not absent in w. By (32) and (15), with $V = U + Ay$,

$$w = \frac{1}{U+Ay} \int_{-\infty}^{y} (U+Ay)\left(\frac{\partial}{\partial y} + \frac{A}{U}\right)\left(\frac{mz}{4\pi r^3}\right) dy, \tag{33}$$

which after an integration by parts gives

$$w = \frac{mz}{4\pi r^3} - \frac{A^2}{U(U+Ay)} \frac{mz}{4\pi r}, \tag{34}$$

of which the last term is a tertiary-flow term. More seriously, this term makes the integration of the equation of continuity to obtain u impossible without allowing u to become logarithmically infinite at both $x = +\infty$ and $x = -\infty$. This is an extremely discouraging feature of the present solution. However, it is related to the circumstance that $V(y)$ actually becomes zero for finite y, and we shall see that for more realistic velocity distributions the difficulty does not arise.

There is one other $V(y)$, not specially realistic, but at least not leading to unrealistic infinities, for which a simple solution is possible. This is

$$V(y) = Ue^{\lambda y}. \tag{35}$$

Again, the solution is most readily obtained from equation (19), which in this case becomes

$$\nabla^2 v - \lambda^2 v = \left(\frac{\partial}{\partial y} + \lambda\right) m\delta(x)\delta(y)\delta(z), \tag{36}$$

whence

$$v = \left(\frac{\partial}{\partial y} + \lambda\right)\left[-\frac{me^{-\lambda r}}{4\pi r}\right], \tag{37}$$

the term in square brackets being the fundamental solution of $\nabla^2 v = \lambda^2 v$. Alternatively, (37) can be obtained by first solving (26) and then making use of the result (Watson 1944, § 13.47 (2) with $\mu = 0$, $\nu = \frac{1}{2}$)

$$\int_0^{\infty} \frac{k}{\sqrt{(k^2+\lambda^2)}} e^{-|y|\sqrt{(k^2+\lambda^2)}} J_0(ks)\, dk = \frac{e^{-\lambda\sqrt{(y^2+s^2)}}}{\sqrt{(y^2+s^2)}}. \tag{38}$$

The modification of the primary- and secondary-flow terms in (37) by the exponential factor inside the square bracket means that tertiary-flow

terms in λ^2 and higher-order terms are present in v in this case, but that they all add up to a solution tending to zero rapidly as $r \to \infty$. As to w, equation (15) with (37) gives simply

$$w = \frac{\partial}{\partial z}\left(-\frac{me^{-\lambda r}}{4\pi r}\right), \tag{39}$$

and for u the equation of continuity yields after some reduction

$$u = \frac{\partial}{\partial x}\left(-\frac{me^{-\lambda r}}{4\pi r}\right) + \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)\int_{-\infty}^{x} \frac{e^{-\lambda r}}{r}\,dx. \tag{40}$$

Thus there is an exponential falling-off as $r \to \infty$ in u also, except as $x \to +\infty$ for values of y and z which are not large. This last result is realistic, and corresponds to the fact that fluid which has passed close to the source has been permanently affected, because its vortex lines have been deformed. In fact,

$$\begin{aligned}\lim_{x\to+\infty}(u) &= \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)\int_{-\infty}^{\infty} \frac{e^{-\lambda\sqrt{(x^2+y^2+z^2)}}}{\sqrt{(x^2+y^2+z^2)}}\,dx \\ &= \frac{m\lambda}{4\pi}\left(\frac{\partial}{\partial y} + \lambda\right)2K_0\{\lambda\sqrt{(y^2+z^2)}\},\end{aligned} \tag{41}$$

by Watson (1944), § 6.22 (5). All the outflow due to the source has somehow been channelled into the region immediately downstream of it*. That the total volume flow in the x-direction in this region is in fact m is easily verified from (41); we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lim_{x\to+\infty} u)\,dydz = m\int_0^{\infty} tK_0(t)\,dt = m, \tag{42}$$

by Watson (1944), § 13.21 (8). The solution, however, is not worth pursuing further because of the impossibility that in a real flow the velocity distribution could be well represented by (35) over any large range of y.

4. Method of solution in the general case

We pass to methods applicable for a general distribution of velocity $V(y)$, subject only to the restriction that the limits $V(\infty)$ and $V(-\infty)$ exist, and that $V(y)$ is nowhere zero, even at these limits. (We shall also require $V(y)$ to tend to these limits sufficiently fast for certain integrals to converge.) Thus, the parallel flow is supposed to become uniform as $y \to +\infty$ and as $y \to -\infty$, as in a wake or in a mixing region between two streams.

The equation (26) as $y \to +\infty$ tends to the simple form (29) and has solutions e^{-ky} and e^{ky}. Only the former tends to zero and so is of interest to us. We therefore introduce the notation $v_1(y, k)$ for the solution of (26) which is asymptotic to e^{-ky} as $y \to +\infty$. Similarly, $v_2(y, k)$ signifies the

* We shall see that this is due to the displacement of streamlines, as they pass the source, towards the region of lower undisturbed velocity.

solution which is asymptotic to e^{ky} as $y \to -\infty$. The necessary discontinuous solution v_0 must take the form

$$v_0 = A(k)v_1(y,k) \quad (y > 0), \qquad B(k)v_2(y,k) \quad (y < 0), \tag{43}$$

for some constants A and B. But, by (27),

$$A(k)v_1(0,k) - B(k)v_2(0,k) = \frac{m}{4\pi^2}, \tag{44}$$

$$A(k)v_1'(0,k) - B(k)v_2'(0,k) = \frac{m}{4\pi^2}\frac{V'(0)}{V(0)}, \tag{45}$$

and the solution of these simultaneous equations is

$$\left.\begin{aligned} A(k) &= \frac{m}{4\pi^2 W(k)}\left\{v_2'(0,k) - \frac{V'(0)}{V(0)}v_2(0,k)\right\}, \\ B(k) &= \frac{m}{4\pi^2 W(k)}\left\{v_1'(0,k) - \frac{V'(0)}{V(0)}v_1(0,k)\right\}, \end{aligned}\right\} \tag{46}$$

where

$$W(k) = v_1(y,k)v_2'(y,k) - v_2(y,k)v_1'(y,k) \tag{47}$$

is independent of y by a well-known property of second-order linear differential equations (and it is actually (47) for $y = 0$ that has been used to derive (46)).

Equation (46) would be useless if the Wronskian $W(k)$ vanished. However, this cannot happen. For it would mean that v_1 and v_2 were simply proportional to one another, so that a *continuous* (and continuously differentiable) solution v_0 existed tending to zero at $y = \pm\infty$. Physically, a stationary wave could then be present without the need for any extraneous disturbance like the source here discussed. This state of affairs is not at all plausible; but the possibility can be dismissed altogether on mathematical grounds, since if such a v_0 existed, it may be shown from the differential equation (26) that

$$\int_{-\infty}^{\infty} \left\{\left(v_0' - \frac{V'}{V}v_0\right)^2 + k^2 v_0^2\right\} dy = 0, \tag{48}$$

so that v_0 must vanish identically (because otherwise the left-hand side would be positive).

The character of the solution $v_1(y,k)$ (and its sister solution $v_2(y,k)$), on which all has been made to depend, will be sufficiently uncovered in what follows by three theories, one for large k, one for small k, and one for general k subject to the restriction that the total variation of $V(y)$ is small compared with its absolute magnitude. In each case the corresponding properties of $v_0(y,k)$, $v(y,s)$, u and w are inferred from the above formulas and those of § 2.

5. Theory for small r

By properties of the Hankel transformation (25) we shall be able to deduce the behaviour of v for small r from the behaviour of v_0 for large k.

To study the latter we solve equation (26) in the form

$$v_0'' - k^2 v_0 = \frac{V''}{V}v_0$$

by 'variation of parameters'. Picking the solution v_1 asymptotic to e^{-ky} as $y \to +\infty$, we obtain

$$v_1(y,k) = e^{-ky}\left\{1 + \int_y^\infty \frac{1-e^{-2k(q-y)}}{2k}\,\frac{V''(q)}{V(q)}\,v_1(q,k)e^{kq}\,dq\right\}. \tag{49}$$

This integral equation is useful for many purposes, in particular for proving by Picard's well-known method the existence and uniqueness of $v_1(y,k)$, and its analyticity as a function of k for $\mathscr{R}\{k\} \geqslant 0$. It also tells us the expansion of $v_1(y,k)$ for large k, giving to a first approximation $v_1 e^{ky} = 1 + O(k^{-1})$ and to a second

$$v_1 = e^{-ky}\left\{1 + \frac{1}{2k}\int_y^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k^2}\right)\right\}. \tag{50}$$

Similarly,
$$v_2 = e^{ky}\left\{1 + \frac{1}{2k}\int_{-\infty}^y \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k^2}\right)\right\}, \tag{51}$$

and so the Wronskian of v_1 and v_2 is

$$W(k) = v_1 v_2' - v_2 v_1' = 2k + \int_{-\infty}^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k}\right), \tag{52}$$

its independence of y being a check on the work. Hence, by (46),

$$\begin{aligned} A &= \frac{m}{4\pi^2}\left\{2k + \int_{-\infty}^\infty \frac{V''(q)}{V(q)}\,dq + O\left(\frac{1}{k}\right)\right\}^{-1} \times \\ &\qquad \times\left\{k + \frac{1}{2}\int_{-\infty}^0 \frac{V''(q)}{V(q)}\,dq - \frac{V'(0)}{V(0)} + O\left(\frac{1}{k}\right)\right\} \\ &= \frac{m}{8\pi^2}\left\{1 - \frac{1}{2k}\int_0^\infty \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\,\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}, \end{aligned} \tag{53}$$

and similarly

$$B = \frac{m}{8\pi^2}\left\{-1 + \frac{1}{2k}\int_{-\infty}^0 \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\,\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}, \tag{54}$$

whence by (43), (50) and (51)

$$v_0 = \frac{m}{8\pi^2}\,e^{-k|y|}\left\{\operatorname{sgn} y - \frac{1}{2k}\int_0^y \frac{V''(q)}{V(q)}\,dq - \frac{1}{k}\,\frac{V'(0)}{V(0)} + O\left(\frac{1}{k^2}\right)\right\}. \tag{55}$$

The author has evaluated also the coefficient of k^{-2} in the expansion of $v_0(y,k)$ for large k, but the only property of this coefficient that tells us anything about the behaviour of v for small r is that it is $O(|y|)$ as $|y| \to 0$; the next term still is one in k^{-3}. Hence, writing $F(y,k)$ for the error term in (55), we have by (25) and (31)

$$\begin{aligned} v = \frac{my}{4\pi r^3} - \left\{\frac{V'(0)}{V(0)} + \frac{1}{2}\int_0^y \frac{V''(q)}{V(q)}\,dq\right\}\frac{m}{4\pi r} + \\ + \frac{m}{4\pi}\int_0^\infty e^{-k|y|}J_0(ks)F(y,k)\,dk, \end{aligned} \tag{56}$$

with
$$|F(y,k)| < \frac{C_1|y|}{1+k} + \frac{C_2}{(1+k)^2}$$

for some C_1 and C_2. The integral in (56) is therefore bounded as $r \to 0$; it is less in modulus than

$$C_1|y| \int_0^\infty \frac{e^{-k|y|}\,dk}{1+k} + C_2 \int_0^\infty \frac{dk}{(1+k)^2}$$

which is bounded, because the first integral is $O(\log|y|^{-1})$. (Actually, the integral in (56) tends to a finite limit $\int_0^\infty F(0,k)\,dk$ as $r \to 0$. However, this limit cannot be evaluated unless $v_0(0,k)$ is known for all k.) Part of the middle term in (56) is also bounded, and so finally

$$v = \frac{my}{4\pi r^3} - \frac{V'(0)}{V(0)} \frac{m}{4\pi r} + O(1) \quad \text{as } r \to 0. \tag{57}$$

Equation (57) shows that the part of v which tends to infinity as $r \to 0$ consists simply of the primary flow and the simple-shear secondary flow. In other words, near the source the variations of velocity and of shear are small enough for the approximations of neglecting the square of the shear and regarding it as uniform to give the right answer.

It follows from (57) and (15) that

$$w = \frac{\partial}{\partial z}\left(-\frac{m}{4\pi r}\right) + O(1); \tag{58}$$

this also is the sum of the primary flow and the simple-shear secondary flow, since the latter has no z-component if the squares of the disturbances are neglected. Finally, the equation of continuity gives

$$u = \frac{\partial}{\partial x}\left(-\frac{m}{4\pi r}\right) - \frac{m}{4\pi} \frac{V'(0)}{V(0)} \frac{y}{y^2+z^2}\left(1+\frac{x}{r}\right) + O(1), \tag{59}$$

the second term being obtained by integrating y/r^3 with respect to x from the lower limit $-\infty$; again, this second term agrees with the small-disturbance approximation to the simple-shear secondary flow (Lighthill 1957 a).

The work of this section gives a check that fluid is truly emerging from the source at the rate m units of volume per unit time, as well as showing up the character of the simple-shear secondary flow solution as a valid second approximation *in the neighbourhood of the source.*

6. Evaluation of v_0 for small k

By a process inverse to that of § 5, the behaviour of v for large r can be deduced from that of v_0 for small k, which is obtained as follows.

A uniformly valid first approximation to the solution $v_1(y,k)$ for small k is

$$v_1 = \frac{V(y)}{V(\infty)} e^{-ky}. \tag{60}$$

To see this, note that when y is large $V(y) \to V(\infty)$, so (60) tends to e^{-ky} as it should, but when y is not large $e^{-ky} \doteqdot 1$ (for small k), and so (60) is

approximately proportional to $V(y)$, which in turn approximately satisfies (26) for small k. These facts suggest the transformation

$$v_1 = \frac{V(y)}{V(\infty)} e^{-ky} u_1, \tag{61}$$

which on substitution in (26) gives the equation

$$\frac{d}{dy}\left(Y e^{-2ky} \frac{du_1}{dy} \right) = kY' e^{-2ky} u_1, \tag{62}$$

where for shortness we have written $V^2(y)$ as $Y(y)$. From (62) and the condition $u_1 \to 1$ as $y \to \infty$ we obtain the integral equation

$$u_1(y) = 1 + k \int_y^\infty \frac{dq}{Y(q)} \int_q^\infty Y'(l) e^{2k(q-l)} u_1(l)\, dl, \tag{63}$$

whence successive approximations to u_1 for small k may be found, the first being $u_1 = 1 + O(k)$, the second

$$u_1 = 1 + k \int_y^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq + O(k^2), \tag{64}$$

and the third

$$u_1 = 1 + k \int_y^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq - k^2 \int_y^\infty \frac{dq}{Y(q)} \times \\ \times \int_q^\infty \frac{\{Y(\infty) - Y(l)\}\{Y(q) + Y(l)\}}{Y(l)}\, dl + O(k^3). \tag{65}$$

Similarly, we put

$$v_2 = \frac{V(y)}{V(-\infty)} e^{ky} u_2, \tag{66}$$

and find that

$$u_2 = 1 + k \int_{-\infty}^y \frac{Y(-\infty) - Y(q)}{Y(q)}\, dq - \\ - k^2 \int_{-\infty}^y \frac{dq}{Y(q)} \int_{-\infty}^q \frac{\{Y(-\infty) - Y(l)\}\{Y(q) + Y(l)\}}{Y(l)}\, dl + O(k^3). \tag{67}$$

Since also

$$u_1' = -k \frac{Y(\infty) - Y(y)}{Y(y)} + \frac{k^2}{Y(y)} \int_y^\infty \frac{\{Y(\infty) - Y(q)\}\{Y(y) + Y(q)\}}{Y(q)}\, dq + O(k^3), \tag{68}$$

and

$$u_2' = k \frac{Y(-\infty) - Y(y)}{Y(y)} - \frac{k^2}{Y(y)} \int_{-\infty}^y \frac{\{Y(-\infty) - Y(q)\}\{Y(y) + Y(q)\}}{V(q)} \times \\ \times dq + O(k^3), \tag{69}$$

we infer that the Wronskian is

$$W(k) = v_1 v_2' - v_2 v_1' = \frac{Y(y)}{V(\infty)V(-\infty)} (u_1 u_2' - u_2 u_1' - 2k u_1 u_2) \\ = \frac{k\{Y(\infty) + Y(-\infty)\}}{V(\infty)V(-\infty)} + \frac{k^2}{V(\infty)V(-\infty)} \times \\ \times \int_{-\infty}^\infty \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{Y(q)}\, dq + O(k^3), \tag{70}$$

where on collecting terms it was found that all those involving y explicitly disappear, which is a check on the result.

Now, by (46) and (66),

$$A = \frac{m}{4\pi^2 W(k)} \frac{V(0)}{V(-\infty)} \{u_2'(0,k) + k u_2(0,k)\}$$

$$= \frac{m}{4\pi^2 W(k)} \frac{V(0)}{V(-\infty)} \Big[k \frac{Y(-\infty)}{Y(0)} - \frac{k^2}{Y(0)} \times$$

$$\times \int_{-\infty}^{0} \{Y(-\infty) - Y(q)\}\, dq + O(k^3) \Big], \quad (71)$$

and similarly

$$B = \frac{m}{4\pi^2 W(k)} \frac{V(0)}{Y(\infty)} \Big[-k \frac{(Y\infty)}{Y(0)} + \frac{k^2}{Y(0)} \int_{0}^{\infty} \{Y(\infty) - Y(q)\}\, dq + O(k^3) \Big], \quad (72)$$

results which with equation (70) give non-zero limits for A and B as $k \to 0$ and give also their derivatives with respect to k at $k = 0$.

By (43), (61), (65), (70) and (71), for $y > 0$,

$$v_0 = \frac{mV(y)e^{-ky}}{4\pi^2 V(0)} \frac{Y(-\infty)}{Y(\infty) + Y(-\infty)} \times$$

$$\times \Big[1 + k \int_{y}^{\infty} \frac{Y(\infty) - Y(q)}{Y(q)}\, dq - k \int_{-\infty}^{0} \frac{Y(-\infty) - Y(q)}{Y(-\infty)}\, dq -$$

$$- k \int_{-\infty}^{\infty} \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{\{Y(\infty) + Y(-\infty)\} Y(q)}\, dq + O(k^2) \Big], \quad (73)$$

and similarly for $y < 0$

$$v_0 = -\frac{mV(y)e^{ky}}{4\pi^2 V(0)} \frac{Y(\infty)}{Y(\infty) + Y(-\infty)} \times$$

$$\times \Big[1 + k \int_{-\infty}^{y} \frac{Y(-\infty) - Y(q)}{Y(q)}\, dq - k \int_{0}^{\infty} \frac{Y(\infty) - Y(q)}{Y(q)}\, dq -$$

$$- k \int_{-\infty}^{\infty} \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{\{Y(\infty) + Y(-\infty)\} Y(q)}\, dq + O(k^2) \Big]. \quad (74)$$

Now, equations (73) and (74) are to be used differently in different regions. For large $|y|$, they are best used as they stand, or at most after such simplification as is gained from replacing $V(y)$ by $V(\infty)$ or $V(-\infty)$, as the case may be, and suppressing the first integral inside each square bracket. However, for values of $|y|$ which are not large, it is permissible, and valuable, to expand the $e^{-k|y|}$ factor outside the bracket in each equation and take it inside. When this is done, both (73) and (74) can be rearranged into a common form in which the coefficient of k is seen

to be a continuously differentiable function of y even at $y = 0$. This form for $|y|$ not large is

$$v_0 = \frac{mV(y)}{4\pi^2 V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)} \left[\frac{\operatorname{sgn} y}{Y(\infty \operatorname{sgn} y)} - k \int_0^y \frac{dq}{Y(q)} + \right.$$
$$+ \frac{k}{Y(\infty)+Y(-\infty)} \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\, dq - \right.$$
$$\left. \left. - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\, dq \right\} + O(k^2) \right]. \quad (75)$$

The term independent of k in (75) is still discontinuous at $y = 0$, but this will be found to produce no discontinuity in v itself, which is influenced only by the term proportional to k.

7. Theory for large r

We now show that the properties of

$$v = 2\pi \int_0^\infty v_0(y,k) k J_0(ks)\, dk \quad (25\text{ bis})$$

for large $r = \sqrt{(y^2+s^2)}$ can be deduced from the equations (73), (74) and (75) which give the behaviour of $v_0(y,k)$ for small k.

The case when $|y|$ is itself large is especially easy and may as well be treated first. For large positive y, (73) shows that

$$v_0 \sim \frac{m_1}{8\pi^2} e^{-ky}(1-kc_1) \quad (76)$$

as $k \to 0$, where

$$m_1 = \frac{V(\infty)}{V(0)} \frac{2V^2(-\infty)}{V^2(\infty)+V^2(-\infty)} m, \quad (77)$$

and

$$c_1 = \int_{-\infty}^0 \frac{Y(-\infty)-Y(q)}{Y(-\infty)}\, dq + \int_{-\infty}^\infty \frac{\{Y(\infty)-Y(q)\}\{Y(-\infty)-Y(q)\}}{\{Y(\infty)+Y(-\infty)\}Y(q)}\, dq. \quad (78)$$

The 'Borel' theory of integrals with exponential factors whose exponent consists of the variable of integration multiplied by a large parameter (here y) tells us that as $y \to \infty$

$$v_1 \sim \frac{1}{4\pi} \int_0^\infty m_1 e^{-ky}(1-kc_1) J_0(ks)\, dk = -\left(m_1 \frac{\partial}{\partial y} + m_1 c_1 \frac{\partial^2}{\partial y^2} \right) \frac{1}{4\pi r}, \quad (79)$$

where (31) has been used. The behaviour of w follows at once from (15) with the lower limit replaced by $+\infty$, in which V can be regarded as constant on both sides, giving

$$w \sim -\left(m_1 \frac{\partial}{\partial z} + m_1 c_1 \frac{\partial^2}{\partial y \partial z} \right) \frac{1}{4\pi r}. \quad (80)$$

The equation of continuity then gives a similar result for u, and the three results may be written

$$(u, v, w) \sim \operatorname{grad}\left(-\frac{m_1}{4\pi r}\right) + c_1 \frac{\partial}{\partial y} \operatorname{grad}\left(-\frac{m_1}{4\pi r}\right). \tag{81}$$

Equation (81) shows that, for large positive y, the effect of the source of strength m at the origin, in the midst of the sheared flow, is the same as if a source of different strength m_1, and a doublet of strength $m_1 c_1$ with its axis in the y-direction, were present at the origin, but immersed in a uniform flow with velocity $V(\infty)$. (These are the two leading terms in the expansion of the irrotational disturbance above the shear layer in terms of singularities at the origin.)

Alternatively, we can say that the effect is the same as if a simple source of strength m_1 were present, not at the origin, but at the displaced position $(0, -c_1, 0)$.

Similarly, for large negative y,

$$(u, v, w) \sim \operatorname{grad}\left(-\frac{m_2}{4\pi r}\right) - c_2 \frac{\partial}{\partial y} \operatorname{grad}\left(-\frac{m_2}{4\pi r}\right), \tag{82}$$

where

$$m_2 = \frac{V(-\infty)}{V(0)} \frac{2V^2(\infty)}{V^2(\infty) + V^2(-\infty)} m, \tag{83}$$

and

$$c_2 = \int_0^\infty \frac{Y(\infty) - Y(q)}{Y(q)}\, dq + \int_{-\infty}^\infty \frac{\{Y(\infty) - Y(q)\}\{Y(-\infty) - Y(q)\}}{\{Y(\infty) + Y(-\infty)\} Y(q)}\, dq. \tag{84}$$

Thus, for large negative y, the disturbance is that which would be produced in a uniform flow by a source of yet different strength m_2 at the origin, and a doublet of strength $m_2 c_2$ with its axis in the negative y-direction. Again, it can if preferred be regarded as equivalent to a source of strength m_2 at the displaced position $(0, +c_2, 0)$.

The strengths m_1 and m_2 of the equivalent sources can alternatively be deduced by an image approach due to Mr M. B. Glauert. To obtain the asymptotic disturbance at large distances, we replace the shear layer by a concentrated vortex sheet. Now, suppose first that the source of strength m is 'above' the vortex sheet, in the stream with velocity $V(\infty)$. By applying the condition of continuous pressure and flow direction across the vortex sheet, it may be shown that if squares of disturbances are neglected the flow in the upper region is equivalent to that due to the original source of strength m and a source of strength m_i (i for image) placed at its mirror-image in the vortex sheet; further, the flow in the lower region is that due to a source of different strength m_t (t for transmitted) at the position of the original source. Here,

$$m_i = \frac{V^2(-\infty) - V^2(\infty)}{V^2(-\infty) + V^2(\infty)} m, \qquad m_t = \frac{2V(\infty)V(-\infty)}{V^2(\infty) + V^2(-\infty)} m. \tag{85}$$

When the original source approaches close to the vortex sheet it almost coincides with its image, so effectively in the upper region we have a source of strength $m + m_i$, and in the lower region one of strength m_t. Finally,

when the source is in the shear layer itself, at $y = 0$, the fluid produced by it fills a cylinder (stretching back from the source) of cross-sectional area $m/V(0)$; this measures how much the rest of the flow is pushed out. Its effect is the same as if a source of strength $mV(\infty)/V(0)$ were present just above the shear layer where the velocity is $V(\infty)$. Therefore, the effective source strengths m_1 and m_2 in the regions above and below the shear layer respectively should be

$$m_1 = (m+m_i)\frac{V(\infty)}{V(0)}, \qquad m_2 = m_t \frac{V(\infty)}{V(0)}, \tag{86}$$

which agree with (77) and (83).

The effective displacements c_1 and c_2 of the source (equations (78) and (84)) can probably not be calculated by such a method, as they are of the order of the thickness of the shear layer itself.

The limiting case when $V(-\infty)/V(\infty) \to 0$ is of interest; then both m_1 and m_2 tend to 0. This is connected with the fact that $m_i \to -m$ (the image of a source in a 'free streamline' separating a flow from a dead-air region being an equal sink), so that the source and its image cancel. They leave as the leading term in the disturbance in the upper region of uniform flow, a doublet whose strength

$$m_1 c_1 \to -\frac{m}{V(0)V(\infty)} \int_{-\infty}^{0} V^2(q)\, dq. \tag{87}$$

The case $V(y) = Ue^{\lambda y}$ studied in § 3 may be compared with this, since that distribution has in a sense $V(-\infty)/V(\infty) = 0$. But for large r (except where y and z are not large and x is positive) the disturbance in that case was found to fall off like $e^{-\lambda r}$, so that not only the effective source but also the effective dipole and higher multipole strengths are zero; possibly the $V(\infty)$ in the denominator in (87) might have led one to expect this.

Note that the net volume flow through the part of the surface of a large sphere, with centre at the origin, which lies outside the shear layer, is

$$\tfrac{1}{2}(m_1+m_2) = \frac{V(\infty)V(-\infty)\{V(\infty)+V(-\infty)\}}{V(0)\{V^2(\infty)+V^2(-\infty)\}}\, m. \tag{88}$$

This differs from m, being less than m for a mixing region and greater for a wake. The balance is made up by flow across the part of the surface of the sphere immediately downstream of the source, where streamlines have been displaced. In a mixing region, for example, the displacement is towards the region of lower velocities, and so the velocity at any fixed point in space is increased. This point will be returned to in § 8.

We begin now to investigate the flow for large r within the shear layer, that is, where $|y|$ is not large and therefore equation (75) holds. This work depends on the result that

$$\int_0^\infty F(k)kJ_0(ks)\, dk \sim -\frac{F'(0)}{s^3} + \frac{3}{2}\frac{F'''(0)}{s^5} - \frac{3.5}{2.4}\frac{F^{\mathrm{v}}(0)}{s^7} + \ldots \tag{89}$$

as $s \to \infty$, if $F(k)$ is a bounded analytic function of k for $0 \leqslant k \leqslant \infty$. The author has been unable to find a clear-cut reference for this result, though

it must presumably be well-known to people who frequently use the Hankel transform. It can be approached either by real- or complex-variable methods, the latter being simpler, though they assume rather more.

If, in fact, $F(k)$ is regular in a wedge $|\arg k| \leqslant \alpha$, then we write

$$J_0(ks) = \frac{K_0(-iks) - K_0(iks)}{\pi i} \tag{90}$$

(by Watson 1944, § 3.7 (8) and § 3.62 (5)), and divide the integral (89) into two parts, shifting the path of integration to $\arg k = +\alpha$ for the part involving $K_0(-iks)$ and to $\arg k = -\alpha$ for the part involving $K_0(iks)$. The integral becomes

$$\frac{1}{\pi i}\int_0^{\infty e^{i\alpha}} F(k)kK_0(-iks)\,dk - \frac{1}{\pi i}\int_0^{\infty e^{-i\alpha}} F(k)kK_0(iks)\,dk$$

$$= \frac{1}{\pi}\int_0^{\infty e^{i(\alpha-\frac{1}{2}\pi)}} F(it)itK_0(ts)\,dt + \frac{1}{\pi}\int_0^{\infty e^{i(\frac{1}{2}\pi-\alpha)}} F(-it)(-it)K_0(ts)\,dt. \tag{91}$$

Since $K_0(ts)$ falls off exponentially along each path in (91), the integrals may be asymptotically expanded, by a method directly analogous to the 'Borel' method used earlier in this section, as

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\frac{i^{n+1}F^{(n)}(0)}{n!}\int_0^{\infty e^{i(\alpha-\frac{1}{2}\pi)}} t^{n+1}K_0(ts)\,dt + \frac{(-i)^{n+1}F^{(n)}(0)}{n!}\times\right.$$

$$\left.\times\int_0^{\infty e^{i(\frac{1}{2}\pi-\alpha)}} t^{n+1}K_0(ts)\,dt\right]. \tag{92}$$

The contour integrals in (92) are clearly identical (and their values are simple combinations of factorials; see Watson 1944, § 13.21 (8)); hence the terms with n even in the sum vanish, and the rest take the form (89).

We wish to apply (89) to the case $F(k) = v_0(y, k)$, so the question of the analyticity as functions of k of the expressions (43) for v_0 arises. The analycitity of $v_1(y, k)$ and $v_2(y, k)$ and their derivatives with respect to y for $\mathscr{R}\{k\} \geqslant 0$ is easily proved by Picard's method from the integral equation (49). Hence v_0 will be analytic provided that $W(k)$ is non-zero. But the proof based on (48) that this is so can easily be extended to complex k with non-negative real part; in fact, if $W(k)$ vanished, so that a continuously differentiable solution of (26) tending to zero as $y \to \pm\infty$ existed, then the integral (48) would still be zero if k^2 were replaced by its real part and all other squares replaced by squares of moduli, and the conclusion that v_0 is identically zero would follow as before provided that $\mathscr{R}\{k^2\} \geqslant 0$. Hence we may take $\alpha = \frac{1}{4}\pi$ in the above discussion.

Equation (89), whose applicability to the case $F(k) = v_0(y, k)$ has thus been established, now gives with (25) and (75) that

$$v = \frac{mV(y)}{2\pi V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)}\left[\int_0^y \frac{dq}{Y(q)} - \frac{1}{Y(\infty)+Y(-\infty)} \times \right.$$
$$\left. \times \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\,dq - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\,dq \right\}\right] \frac{1}{s^3} + $$
$$+ O\left(\frac{1}{s^5}\right), \quad (93)$$

as $s = (x^2+z^2)^{1/2} \to \infty$ for values of y which are not large.

It is, perhaps, tiresome that the expression for v when r is large should take three different forms (79), (82) and (93), according as y is large and positive, large and negative, or not large at all, and it is natural to ask whether they can all be replaced by a single, uniformly valid asymptotic form. This does not seem to be possible. By careful inspection of (79) and (93) one can show that both are included in the more general expression

$$v = \frac{V(y)}{V(\infty)}\left\{ -\left(m_1 \frac{\partial}{\partial y} + m_1 c_1 \frac{\partial^2}{\partial y^2}\right)\frac{1}{4\pi r} - \frac{m_1}{4\pi r^3}\int_y^\infty \frac{Y(\infty)-Y(q)}{Y(q)}\,dq \right\} + O\left(\frac{1}{r^5}\right), \quad (94)$$

which therefore is valid both when y is large and positive and when $|y|$ is not large; but for large negative y (94) tends to

$$\frac{m_2 y}{4\pi r^3} - \frac{m_2 c_2}{4\pi r^3} \quad (95)$$

whereas by (82) there should be an additional term $3m_2c_2y^2/4\pi r^5$ for large negative y.

However, such an expression as (94) is just what one needs to integrate (15) (with lower limit $+\infty$) to obtain an expression for w when s but not $|y|$ is large. This gives simply

$$w = \frac{2Y(\infty)Y(-\infty)}{V(y)V(0)\{Y(\infty)+Y(-\infty)\}} \frac{mz}{4\pi s^3} + O\left(\frac{1}{s^4}\right), \quad (96)$$

whence, by (93) and the equation of continuity,

$$u = \frac{2Y(\infty)Y(-\infty)}{V(y)V(0)\{Y(\infty)+Y(-\infty)\}} \frac{mx}{4\pi s^3} - \frac{mV'(y)}{2\pi V(0)} \frac{Y(\infty)Y(-\infty)}{Y(\infty)+Y(-\infty)} \times$$
$$\times \left[\int_0^y \frac{dq}{Y(q)} - \frac{1}{Y(\infty)+Y(-\infty)} \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\,dq - \right.\right.$$
$$\left.\left. - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\,dq \right\}\right] \frac{1+x/s}{z^2} + O\left(\frac{1}{s^4}\right). \quad (97)$$

The leading terms of (96) and (97) show how the effective source strength varies *inversely* as the undisturbed flow velocity $V(y)$ within the shear layer, tending to m_1 (see (77)) as $y \to +\infty$ and to m_2 (see (83)) as $y \to -\infty$. This is connected with the fact that the pressure is constant across the shear layer far from the source, and that the pressure varies like $-\rho V(y)u$

along any streamline. The additional term in u, not inversely proportional to $V(y)$, is due to the displacement of the streamlines.

8. Theory for shear layers with small total variation of velocity

In § 4 to § 7 the theory has been developed for a general shear layer, and asymptotic expressions for the velocity field found, both as $r \to 0$ (in the form of terms of orders r^{-2} and r^{-1} plus an error of order 1), and as $r \to \infty$ (in the form of terms of order r^{-2} and r^{-3} plus an error of order r^{-4}). However, the detailed characteristics of the join between these solutions remain unknown, and it was to give some understanding of these that the solution to be described in this section was sought. This solution is valid for all values of x, y and z, but only for a restricted class of shear layers, namely those within which the total variation of the undisturbed velocity $V(y)$ is small compared with $V(y)$ itself. If the minimum value of $V(y)$ is $(1-\epsilon)$ times the maximum, we seek an approximate solution by consistently neglecting ϵ^2.

As before, we begin by obtaining $v_1(y,k)$. The integral equation (49) shows that, if the first power of ϵ be neglected, v_1 is simply e^{-ky}. This first approximation is valid uniformly, even for small k, since the fraction involving k within the integral is always less than $(z-y)$; and indeed the solution (60), valid in the limit $k \to 0$, differs from e^{-ky} only by a term of order ϵ.

Applying equation (49) again, we now see that

$$v_1(y,k) = e^{-ky}\left\{1 + \int_y^\infty \frac{1-e^{-2k(q-y)}}{2k}\,\frac{V''(q)}{V_m}\,dq\right\} + O(\epsilon^2). \tag{98}$$

Equation (98) is similar to the earlier approximation (51), except that the exponential factor cannot now be suppressed because k is not being assumed large. However, the $V(q)$ in the denominator can be replaced by a mean value V_m (say, half the sum of the maximum and minimum of $V(y)$) with an error $O(\epsilon^2)$.

Similarly,

$$v_2(y,k) = e^{ky}\left\{1 + \int_{-\infty}^y \frac{1-e^{2k(q-y)}}{2k}\,\frac{V''(q)}{V_m}\,dq\right\} + O(\epsilon^2), \tag{99}$$

and also

$$\left.\begin{aligned} v_1' &= e^{-ky}\left[-k - \frac{1}{2}\int_y^\infty \left\{1+e^{-2k(q-y)}\right\}\frac{V''(q)}{V_m}\,dq\right] + O(\epsilon^2), \\ v_2' &= e^{ky}\left[k + \frac{1}{2}\int_{-\infty}^y \left\{1+e^{2k(q-y)}\right\}\frac{V''(q)}{V_m}\,dq\right] + O(\epsilon^2), \end{aligned}\right\} \tag{100}$$

whence it follows easily that the Wronskian $v_1 v_2' - v_2 v_1'$ is simply

$$W(k) = 2k + \int_{-\infty}^\infty \frac{V''(q)}{V_m}\,dq + O(\epsilon^2) = 2k + O(\epsilon^2). \tag{101}$$

Next, by (46),

$$\left.\begin{aligned} A(k) &= \frac{\pi}{8\pi^2 k}\left\{k+\frac{1}{2}\int_{-\infty}^{0}(1+e^{2kq})\frac{V''(q)}{V_m}\,dq-\frac{V'(0)}{V_m}+O(\epsilon^2)\right\},\\ B(k) &= \frac{m}{8\pi^2 k}\left\{-k-\frac{1}{2}\int_{0}^{\infty}(1+e^{-2kq})\frac{V''(q)}{V_m}\,dq-\frac{V'(0)}{V_m}+O(\epsilon^2)\right\}. \end{aligned}\right\} \tag{102}$$

Hence, by (43), (98), (99) and (102), if ϵ^2 is neglected,

$$v_0(y,k) = \frac{m}{8\pi^2}\left\{e^{-k|y|}\operatorname{sgn} y-\frac{V'(0)+V'(y)}{2V_m}\,\frac{e^{-k|y|}}{k}-\frac{1}{2k}\times\right.$$
$$\left.\times\left(\int_{y}^{\infty}-\int_{-\infty}^{0}\right)e^{-k|2q-y|}\,\frac{V''(q)}{V_m}\,dq\right\}. \tag{103}$$

Equation (103) has been written in the form which gives clearest confirmation of the result proved in § 5, that v differs from the sum of the primary flow and the simple-shear secondary flow by an expression bounded as $r\to 0$. For it gives the Hankel transform (25), after the substitution $|2q-y| = 2l$ in the last integral, and use of the result (31), as

$$v = \frac{my}{4\pi r^3}-\frac{V'(0)+V'(y)}{2V_m}\,\frac{m}{4\pi r}-\frac{m}{8\pi V_m}\times$$
$$\times\int_{\frac{1}{2}|y|}^{\infty}\frac{V''(\frac{1}{2}y+l)-V''(\frac{1}{2}y-l)}{\sqrt{(4l^2+s^2)}}\,dl. \tag{104}$$

As $r\to 0$, this satisfies

$$v = \frac{my}{4\pi r^3}-\frac{V'(0)}{V_m}\,\frac{m}{4\pi r}-\left\{\frac{V''(0)y}{2V_m}\,\frac{m}{4\pi r}+\frac{m}{16\pi V_m}\int_{0}^{\infty}\frac{V''(l)-V''(-l)}{l}\,dl\right\}+o(1). \tag{105}$$

The terms in curly brackets in (105) may be called the 'tertiary flow' contribution to v; they are the limit as $r\to 0$ of what remains when the primary-flow and the simple-shear secondary-flow terms are taken away. The presence of the integral shows that this tertiary flow depends on the value of $V''(y)$ not only at the source itself but throughout the shear layer.

Expression (104) is not, however, the most suitable form of v for showing its behaviour away from the source. For this, we first integrate (103) by parts, substitute $2q = l$ and only then take its Hankel transform, namely

$$v = \frac{my}{4\pi r^3}+\frac{m}{4\pi}\left(\int_{2y}^{\infty}-\int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\,\frac{y-l}{\{(y-l)^2+s^2\}^{3/2}}\,dl. \tag{106}$$

(Alternatively, on integrating (104) by parts and changing the variable of integration we get (106).)

Equation (106) has a simple and valuable geometrical interpretation. It says that the y-component of disturbance velocity v is that due to the source of strength m at the origin, together with a line distribution of sources along certain stretches of the y-axis; to be precise, the stretches from $2y$ to ∞ and from $-\infty$ to 0 when y is positive, or $\left(\text{since } \int_{2y}^{\infty}-\int_{-\infty}^{0} \text{ is the same as } \int_{0}^{\infty}-\int_{-\infty}^{2y}\right)$ the stretches from 0 to ∞ and from $-\infty$ to $2y$ when y is

negative. The source strength at the point $(0, l, 0)$ is $V'(\frac{1}{2}l)\operatorname{sgn} l/2V_m$ per unit length.

The values of w and u can be interpreted in the same way. Thus, by (15),

$$w = \frac{\partial}{\partial z}\left\{\frac{1}{V(y)}\int_{\infty}^{y} V(q)v(x,q,z)\,dq\right\}$$
$$= \frac{\partial}{\partial z}\left\{\int_{\infty}^{y} v(x,q,z)\,dq + \int_{\infty}^{y}\frac{V(q)-V(y)}{V_m}\frac{mq}{4\pi(q^2+s^2)^{3/2}}\,dq\right\} \qquad (107)$$

if $O(\epsilon^2)$ be neglected, and when the integration is carried out we find that

$$w = \frac{mz}{4\pi r^3} + \frac{m}{4\pi}\left(\int_{2y}^{\infty} - \int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\frac{z}{\{(y-l)^2+s^2\}^{3/2}}\,dl, \qquad (108)$$

which is the z-velocity due to the distribution of sources described above. Also, on integrating

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \qquad (109)$$

from $-\infty$ to x to obtain u, we clearly get the x-velocity due to the same distribution of sources, except from one extra term in $\partial v/\partial y$ resulting from the dependence on y of one of the limits of integration in (106). This additional term on the right of (109) makes a contribution

$$-\int_{-\infty}^{x}\frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{y}{(x^2+y^2+z^2)^{3/2}}\,dx \qquad (110)$$

to u, whence (by evaluating this integral)

$$u = \frac{mx}{4\pi r^3} + \frac{m}{4\pi}\left(\int_{2y}^{\infty} - \int_{-\infty}^{0}\right)\frac{V'(\frac{1}{2}l)}{2V_m}\frac{x}{\{(y-l)^2+s^2\}^{3/2}}\,dl -$$
$$- \frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{y(1+x/r)}{y^2+z^2}. \qquad (111)$$

Thus, apart from the last term in (111), the complete system of disturbance velocities can be identified with the flow due to the source distribution along the y-axis described above.

This does not mean that it is irrotational. The velocity at a point (x, y, z) is that due, not to a fixed system of sources, but to one which itself varies with y. This gives additional terms in the y-derivatives of u and w which (together with the y- and z-derivatives of the extra term in u) prevent any component of the vorticity from vanishing. An actual expression for the vorticity is obtained later (equation (117), in which by (120) ϕ must be taken as $-m/4\pi r V_m$).

Mr M. B. Glauert has shown the author an ingenious derivation of the equivalent source distribution from his image theory mentioned in § 7. He regards the shear layer as made up of a very large number of elementary vortex sheets. The velocity at a point (x, y, z) due to the source at the origin is then represented as the velocity field of the original source itself, together with its images in all vortex sheets which do not pass *between* the origin and (x, y, z). This is permissible if $O(\epsilon^2)$ be neglected because (i) the 'transmitted' source strength m_t, by which the original source strength m needs to be replaced as a result of vortex sheets between the

origin and (x, y, z), is equal to $m+O(\epsilon^2)$ (as (85) shows) because the difference of velocities across all these vortex sheets is $O(\epsilon)$, and (ii) the images of the image sources will have strength $O(\epsilon^2)$.

It follows that there will be image sources from $2y$ to ∞ and from $-\infty$ to 0 if $y > 0$, and from $-\infty$ to $2y$ and from 0 to ∞ if $y < 0$. Further, the source strength between $(0, l, 0)$ and $(0, l+dl, 0)$ due to the vortex sheet lying between $(0, \frac{1}{2}l, 0)$ and $(0, \frac{1}{2}l+\frac{1}{2}dl, 0)$, across which the velocity change is $V'(\frac{1}{2}l)\frac{1}{2}\,dl$, is seen from (85) to be $\{V'(\frac{1}{2}l)\operatorname{sgn} l/2V_m\}\,dl$. The source distribution is therefore exactly that obtained analytically.

To explain the additional term in (111), note that Glauert's vortex sheets are displaced by the source flow. To a first approximation (neglecting $O(\epsilon)$) the y-component of displacement is

$$\frac{1}{V_m}\int_{-\infty}^{x}\frac{my}{4\pi r^3}\,dx = \frac{my(1+x/r)}{4\pi V_m(y^2+z^2)}. \tag{112}$$

The disturbance velocity u will therefore have an additional component approximately

$$V\left\{y-\frac{my(1+x/r)}{4\pi V_m(y^2+z^2)}\right\}-V(y) \doteqdot -\frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{y(1+x/r)}{y^2+z^2}, \tag{113}$$

as in (111), with error $O(\epsilon^2)$.

The equivalent source distributions appropriate to a point with $y > 0$ and to one with $y < 0$ are illustrated for a mixing-region flow in figure 1. A general picture of the disturbance flows can be readily got from this figure. If y were actually greater than the greatest value it takes in the shear layer, the upper part of the equivalent source distribution would disappear altogether. In this region, therefore, the flow is the irrotational flow due to a fixed system of sources. Their total strength is easily verified to be m_1, and their total dipole moment $m_1 c_1$ (see § 7), to within an error $O(\epsilon^2)$. Similar results hold for values of y less than the least value it takes in the shear layer. Within the shear layers, the sources and sinks are so arranged as to produce a pronounced 'downwash' in the case of a mixing region, but this is somewhat mitigated in the case of a wake.

Far downstream of the source, where x is large and positive but y and z are not large, the disturbance velocities are dominated by the last term in (111) (interpreted in (113) as a displacement term); thus,

$$(u, v, w) \sim \left\{-\frac{m}{4\pi}\frac{V'(y)}{V_m}\frac{2y}{y^2+z^2},\ 0,\ 0\right\}. \tag{114}$$

We may use (114) to check conservation of volume flow. The net volume flow, across that part of a large sphere (with centre the origin) which intersects the portion of shear layer downstream of the source where (114) holds, is

$$-\frac{m}{2\pi V_m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}V'(y)\frac{y}{y^2+z^2}\,dy\,dz = -\frac{m}{2V_m}\int_{-\infty}^{\infty}V'(y)\frac{y\,dy}{|y|}$$
$$= -\frac{m}{2V_m}\{V(\infty)+V(-\infty)-2V(0)\} = m-\tfrac{1}{2}(m_1+m_2), \tag{115}$$

where (88) has been used with $O(\epsilon^2)$ neglected. Together with the volume flow $\frac{1}{2}m_1$ across the part of the sphere above the shear layer, and $\frac{1}{2}m_2$ across the part below, this makes up exactly the rate m at which fluid is being produced at the source.

We here have exemplified from the case of small velocity spread how displacement of streamlines in the shear layer can remedy deficiencies between the rate of fluid output at the source and the rate of escape to regions far from the shear layer. In layers with large velocity spread the nature of the displacement is greatly altered (it includes secondary downwash and the like as well as the primary flow due to the source) but it must still produce this same effect. Of course, if terms of the order of the square of the source strength were included, the secondary trailing vorticity would be present, and then the streamlines far downstream would spiral about their mean positions; however, the conclusion would be unaltered.

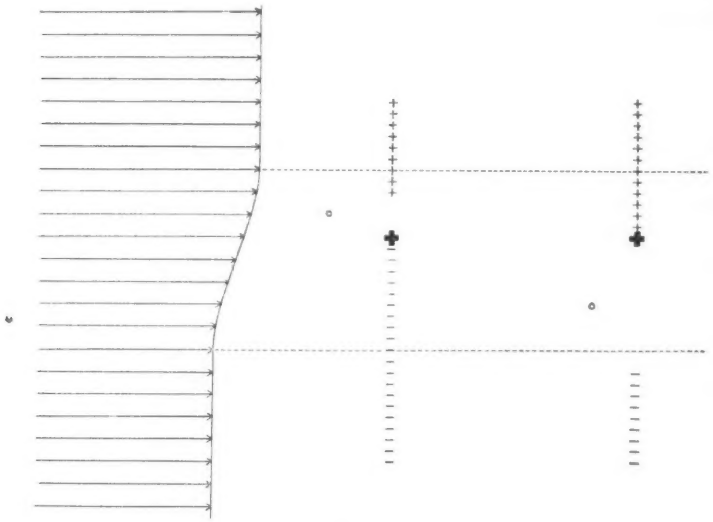

Figure 1. The equivalent source distribution, representing the effect that a weak source in a mixing region with small velocity spread makes at the point marked with a circle (which has $y > 0$ for the left-hand of the two distributions shown, and $y < 0$ for the right-hand one). The heavy plus sign represents the original source in each case, and the light plusses and minuses additional sources and sinks respectively. These can be regarded (Glauert) as images of the original source in the component vortex sheets of the shear layer.

9. Relation between the present solution and the exact-profile secondary flow

In the present paper the disturbance due to a weak source in a shear layer has been investigated by neglecting the square of the disturbance due to the presence of the source, no other approximation being made

in §2 to §7. In §8 the additional approximation of neglecting the square of ϵ (the relative spread of velocity in the shear layer) was made.

It is interesting to compare the results of the double approximation with those of another procedure, in which first the effect of a small non-uniformity in the oncoming stream on a basic irrotational flow (such as that due to the source) is considered by neglecting squares of ϵ, and the result is then further approximated by assuming small disturbances. This procedure was expounded briefly for the case of a general parallel flow in Lighthill (1956, p. 39), although the rest of that paper was devoted to the case of a uniformly sheared upstream flow.

The term 'secondary flow' may appropriately be used to describe any flow field computed by taking an irrotational flow with uniform upstream velocity as the first approximation (or primary flow), and going to a second approximation by allowing a small spread in the upstream velocity but neglecting its square. The secondary flow may be called a simple-shear secondary flow if the upstream flow is taken as uniformly sheared, and an exact-profile secondary flow if the true upstream velocity profile is used.

As shown in Lighthill (1956), secondary flows can be calculated by finding how the vortex lines of the upstream parallel flow become stretched and rotated by the primary flow. The secondary vorticity field having been found, some fairly arduous integrations are needed in general to determine the secondary velocities; but such integrations have been carried out for the simple-shear secondary flow about a sphere in Lighthill (1957 b).

The secondary vorticity field for any upstream velocity profile can be expressed in terms of the 'drift function' t for the primary flow. This function is such that planes of fluid initially at right angles to the stream are distorted by the primary flow into shapes given by equations $t = \text{constant}$; t represents the time at which a given fluid particle reaches a point, measured from when it would have reached $x = 0$ had the uniform upstream flow continued undisturbed. By considering how this distortion of planes of fluids affects the vortex lines of the secondary flow, which initially lie in such planes, one obtains the fairly simple equations (30) of Lighthill (1956) for the exact-profile secondary vorticity field.

These can be further simplified in the region away from the source of disturbance, where disturbances are themselves small, since in this region the drift function takes the simple form

$$t \sim \frac{x-\phi}{U}, \tag{116}$$

where ϕ is a 'disturbance potential', defined so that the full velocity potential of the primary flow is $U(x+\phi)$. The form of the secondary vorticity distribution in this region is easily deduced (Lighthill 1956, (32)) in terms of ϕ; in the special case when the upstream velocity is $V(y)$ (independent of z) it becomes

$$\operatorname{curl}(u, v, \dot{w}) \sim \left\{ -V'(y)\frac{\partial\phi}{\partial z}, \quad -V'(y)\int_{-\infty}^{x} \frac{\partial^2\phi}{\partial y \partial z}\,dx, \right.$$
$$\left. V''(y)\int_{-\infty}^{x} \frac{\partial\phi}{\partial y}\,dx - V'(y)\int_{-\infty}^{x} \frac{\partial^2\phi}{\partial z^2}\,dx \right\}. \tag{117}$$

The disturbance velocity field associated with this disturbance vorticity field is

$$(u, v, w) = \left\{ -V'(y) \int_{-\infty}^{x} \frac{\partial \phi}{\partial y} dx, \quad V'(y)\phi, \quad 0 \right\} - \operatorname{grad} \psi, \tag{118}$$

where

$$\nabla^2 \psi = V''(y)\phi, \tag{119}$$

since the curl of (118) is (117) and its divergence vanishes (as was secured by subtracting grad ψ). In some cases it is necessary to add on to (118) an irrotational part representing the far field of the 'secondary trailing vorticity' (horseshoe vortices wrapped round the disturbance); this part is given in equation (85) of Lighthill (1957 a).

When the disturbance is that due to a source, we may take

$$\phi = -\frac{m}{4\pi r U}. \tag{120}$$

The asymptotic form (118) of the secondary flow for large r then needs no addition due to the far field of the second trailing vorticity, because the latter is of smaller order, $O(r^{-2})$, as $r \to \infty$. Hence in particular

$$v \sim V'(y)\phi - \frac{\partial \psi}{\partial y} \tag{121}$$

as $r \to \infty$.

Now, the solution (121) for the small-disturbance form of the exact-profile secondary flow, which is its asymptotic form far from the source, can be identified with the solution of § 8, where the small disturbances due to a source are approximated for an upstream flow with small velocity spread. For, if $O(\epsilon^2)$ be neglected, equation (19) (except at the origin) becomes

$$\nabla^2 v = \frac{V''(y)}{V_m} \frac{my}{4\pi r^3}, \tag{122}$$

and it is a simple matter to verify that the solution (106) does indeed satisfy (122). Hence, if we were to write v in the form (121), where ϕ is given by (120), then ψ must satisfy

$$\frac{\partial}{\partial y}(\nabla^2 \psi) = V'''(y)\phi + 2V''(y)\frac{\partial \phi}{\partial y} - \nabla^2 v \tag{123}$$

$$= V'''(y)\phi + V''(y)\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\{V''(y)\phi\}, \tag{124}$$

and integrating with respect to y we obtain equation (119) for ψ. Similarly, the identity of w and u in the two solutions may be checked.

However, though this shows that the solution of § 8 could be obtained by the alternative method leading to (118) and (119), the method adopted in this paper seems preferable, because it gives a lot of information about the solution to the more general problem in which ϵ^2 is not neglected, and because the direct solution of the Poisson equation (119) would in any case involve the same Fourier and other analysis which had to be introduced above.

10. Relation to theories of the displacement of the stagnation streamline

One interesting property of shear flow is that when a Pitot tube is placed in such a flow with the open end of the tube pointing upstream the pressure measured in the tube is the stagnation pressure not on the streamline approaching the tube directly along its axis but on one displaced by a measurable amount in the direction of higher velocities (Young & Maas 1936). This may be regarded as evidence of the presence of 'downwash' (flow at right angles to the undisturbed streamlines, in the direction of decreasing velocity) on the streamlines approaching the tube along its axis, and so it becomes a problem to compute this downwash from secondary-flow or other theories and calculate the displacement from it. This is the subject of a companion paper (Lighthill 1957 b), but there is one aspect of it which must be mentioned here, as it can be properly treated only by the present theory. This aspect is the behaviour of the downwash far ahead of the tube.

The 'downwash function' $D(s)$ is the value of $(-v)$ on the axis of the tube ($y = z = 0$) at $x = -s$, and is expected to be positive if $V'(0) > 0$. The most important region of downwash is near the tube; here $D(s)$ is best found from the simple-shear secondary flow, or from a higher approximation in a sequence starting from this. But the downwash so calculated falls off (see (57)) only as

$$D(s) \sim \frac{V'(0)}{V(0)} \frac{m}{4\pi s} \tag{125}$$

at points far enough ahead of the tube both for the disturbances to be small and for the tube to be representable by a source of strength

$$m = \tfrac{1}{4}\pi d_e^2 V(0), \tag{126}$$

where d_e is its external diameter. But the displacement calculated from (125) would be logarithmically infinite, since (125) is not integrable up to $s = \infty$. If tertiary-flow terms were included, they would be even greater for large s, and still worse infinities would arise in the displacement.

The difficulty can be resolved by using the complete solution of this paper in the region where s is large enough for the disturbance to be small and for the tube to be representable by a source of strength (126). In this region the downwash falls off only initially like (125) (see (57)), and later like s^{-3}; more precisely, by (93), we have

$$D(s) \sim \frac{m}{2\pi s^3} \frac{Y(\infty)Y(-\infty)}{\{Y(\infty)+Y(-\infty)\}^2} \left\{ \int_0^\infty \frac{Y^2(\infty)-Y^2(q)}{Y(\infty)Y(q)}\, dq - \right.$$
$$\left. - \int_{-\infty}^0 \frac{Y^2(-\infty)-Y^2(q)}{Y(-\infty)Y(q)}\, dq \right\} \tag{127}$$

as $s \to \infty$. The difficulty of $D(s)$ not being integrable up to $s = \infty$ then disappears.

The problem is thus treated by means of two overlapping solutions. For small to moderate r, the simple-shear secondary-flow theory is used; for moderate to large r the exact-profile small-disturbance theory is used.

Expression (125) is valid in the overlap range of 'moderate' r; for it represents the asymptotic behaviour of the simple-shear secondary-flow theory as $r \to \infty$ and also that of the exact-profile small-disturbance theory as $r \to 0$.

To apply this to the calculation of displacement, let s_0 be a value of s in this overlap region. Then the displacement of the dividing streamline due to downwash $D(s)$ for $s < s_0$ is calculated from the simple-shear secondary flow. For $s > s_0$ the disturbances are small, and the contribution to the displacement from downwash in this region is

$$\frac{1}{V(0)} \int_{s_0}^{\infty} D(s)\, ds, \tag{128}$$

where $D(s)$ is given by the theory of this paper.

Now, we can write the contribution (128) as

$$\frac{1}{V(0)} \int_{s_0}^{s_c} \frac{V'(0)}{V(0)} \frac{m}{4\pi s}\, ds, \tag{129}$$

provided that

$$\frac{V'(0)}{V(0)} \frac{m}{4\pi} \log s_c = \lim_{s_c \to \infty} \left\{ \int_{s_0}^{\infty} D(s)\, ds + \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log s_0 \right\}, \tag{130}$$

and in the limit on the right the downwash function $D(s)$ is taken to be that given by the small-disturbance theory right down to $s_0 = 0$. It is because *this* $D(s)$ is asymptotic to (125) as $s \to 0$ that no substantial difference is made by allowing s_0 to become smaller and vanish in (130), since s_0 has already been assumed sufficiently small for the approximation (125) to be good.

We call s_c, defined by (130), the upstream cut-off of the simple-shear secondary flow. The displacement for $s > s_0$, as (129) shows, is equal to what it would be if the downwash were equal to the simple-shear secondary-flow value (125) up to $s = s_c$ and zero for $s > s_c$—in other words, if the secondary flow were cut off at $s = s_c$.

A precise expression for the cut-off value s_c can be obtained if the theory of § 8 for the case of a small velocity spread be used. Then, by (104),

$$D(s) = \frac{V'(0)}{V(0)} \frac{m}{4\pi s} + \frac{m}{8\pi V_m} \int_0^{\infty} \frac{V''(l) - V''(-l)}{\sqrt{(4l^2 + s^2)}}\, dl. \tag{131}$$

Hence

$$\begin{aligned} \int_{s_0}^{\infty} D(s)\, ds &= \lim_{s_1 \to \infty} \int_{s_0}^{s_1} D(s)\, ds \\ &= \lim_{s_1 \to \infty} \left\{ \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log \frac{s_1}{s_0} + \frac{m}{8\pi} \int_0^{\infty} \frac{V''(l) - V''(-l)}{V_m} \times \right. \\ &\qquad \left. \times \left(\sinh^{-1} \frac{s_1}{2l} - \sinh^{-1} \frac{s_0}{2l} \right) dl \right\} \\ &= \frac{V'(0)}{V(0)} \frac{m}{4\pi} \log \frac{1}{s_0} + \frac{m}{8\pi} \int_0^{\infty} \frac{V''(l) - V''(-l)}{V_m} \left(\log \frac{1}{l} - \sinh^{-1} \frac{s_0}{2l} \right) dl, \end{aligned} \tag{132}$$

whence, by (130),

$$\log s_c = \frac{1}{2V'(0)} \int_0^\infty \{-V''(l)+V''(-l)\}\log l \, dl. \tag{133}$$

Since the denominator $2V'(0)$ in (133) is the value of the integral if the $\log l$ in it be suppressed, we can say that $\log s_c$ is the average value of $\log l$ in the shear layer, if the 'weight' used in averaging is $-V''(l)\operatorname{sgn} l$. This makes s_c itself a 'geometric' mean of l in the shearing layer. (Here, l means the difference in y-coordinate between a general point and the source.) Thus, s_c is of the order of the layer width, as indeed is easy to see geometrically from the 'equivalent source distribution' described in § 8. In more general problems it is reasonable to suppose that s_c is still of the order of the layer width.

We may consider also the implications of the present theory for problems like the displacement effect of a sphere, as treated by Hall (1956) and by Lighthill (1957 b) on the assumption that the inviscid flow is a good enough approximation (in other words, that separation and wake flow do not influence the displacement). The upstream effect of the sphere is that of a doublet of strength

$$m = 2\pi a^3 V(0), \tag{134}$$

if the sphere has radius a and centre the origin. With this value of m, the flow in the region of small disturbances (due to the doublet) is given simply by taking the x-derivative of the velocity field of this paper.

The upstream cut-off of the simple-shear secondary flow is not crucial in this case, since the simple-shear secondary downwash is itself integrable up to $s = \infty$. However, the cut-off reduces the displacement below that obtained by this integration, and the reduction can be calculated, as follows.

If $D_s(s)$ is the downwash function due to the source studied hitherto in this paper, and $D_d(s) = -D_s'(s)$ is that due to the doublet, then the displacement for $s > s_0$ is

$$\begin{aligned}\frac{1}{V(0)} \int_{s_0}^\infty D_d(s)\, ds &= \frac{1}{V(0)} D_s(s_0) \\ &\doteq \frac{1}{V(0)} \left\{ \frac{V'(0)}{V_m} \frac{m}{4\pi s_0} + \frac{m}{16\pi V_m} \int_0^\infty \frac{V''(l)-V''(-l)}{l}\, dl \right\},\end{aligned} \tag{135}$$

where (105) has been used to approximate to $D_s(s_0)$. Now, the first term in curly brackets is the simple-shear secondary-flow value, and so the difference, between the true displacement and that given by assuming that the simple-shear secondary flow persists to infinity, is given approximately (using the value (134) for m) by

$$\frac{a^3}{8V_m} \int_0^\infty \frac{V''(l)-V''(-l)}{l}\, dl. \tag{136}$$

This is presumably negative when $V'(0) > 0$, since the numerator of the integrand is then negative on the average. Thus the correction (136) due to the cut-off is a reduction, as expected.

Actually, it was shown in Lighthill (1957 a) that as well as the doublet of strength (134) the upstream influence of the sphere includes also a rather smaller contribution due to the 'secondary trailing vorticity'. This is equivalent to the effect of a line of doublets with axis the y-axis stretched all along the positive x-axis. However, it may be shown that the displacement resulting from this line of doublets is estimated correctly even if the cut-off studied above is neglected, and hence the validity of the correction (136) remains unaffected. For the practical significance of this correction, see Lighthill (1957 b).

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Skin friction on a strip of finite width moving parallel to its length

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SUMMARY

A flat strip, of infinitesimal thickness and infinite length, is located in a viscous incompressible fluid, and both the strip and fluid are at rest initially. The strip is abruptly set into steady motion parallel to its length. An unsteady uni-directional flow of the fluid results, and there is a variable skin friction on the strip which must be overcome to maintain its velocity. In the early stages of motion, the skin friction is large, with a local behaviour which resembles that for a flat strip of infinite width. The skin friction near each edge of the strip can be more accurately represented by referring to the semi-infinite configuration that is realized on displacement of the other edge to infinity. At this stage, the results depend on separate consideration of the two limiting configurations, and the way to further improvements is not clearly delineated. The object of this paper is to provide a formulation which contains the preceding information and allows a systematic evaluation of all additional refinements. Thus, it is shown that the total skin friction on a strip of width $2a$, moving with velocity V in a fluid whose coefficients of viscosity and kinematic viscosity are μ, ν, takes the form

$$D = 2\mu V\left[\frac{2a}{\sqrt{(\pi\nu t)}} + 1 - \frac{2}{\pi}\int_1^\infty \frac{1}{u^2}\sqrt{\left(\frac{u-1}{u+1}\right)}\operatorname{erfc}\left(\frac{au}{\sqrt{(\nu t)}}\right)du\right].$$

Here the first two terms stem from the infinite and semi-infinite strip distributions, and the integral, containing a complementary error function, furnishes all corrections of the lowest exponential order, in the initial stages of motion, when $a/\sqrt{(\nu t)} \gg 1$. The calculation is based on a new integral equation which makes explicit the effect of an isolated edge, and by iteration provides the interaction effects between edges at a finite separation.

1. INTRODUCTION

A simple class of viscous incompressible fluid motions is that produced by the forced motion of a solid parallel to its infinite length. If the solid is instantaneously set in motion with constant velocity, and has a uniform aspect along its length, the resulting fluid motion is uni-directional, although unsteady in time. The fluid velocity is parallel to and uniform in the direction of motion of the solid, and hence described by a linear

partial differential equation with the time and transverse coordinates as independent variables*. Rayleigh (1911) analysed the simplest configuration, a flat strip of infinite width, for which the skin friction is uniform and has the magnitude $\mu V/\sqrt{(\pi\nu t)}$ per unit area (of each face). The strip of semi-infinite width and the more general wedge shape have been studied by Howarth (1950), Sowerby (1951), and Hasimoto (1951 a, b), and from these calculations the edge effect can be isolated. A characteristic dimension is introduced by the strip of finite width, say $2a$, and the state of affairs depends on whether $a/\sqrt{(\nu t)}$ is large or small compared with unity. In the former case, the fluid motion is confined to a boundary layer with small thickness relative to the strip width, and the total skin friction is given approximately by supplementing the Rayleigh contribution with those of the isolated edges, whence

$$D = \mu V\left[\frac{4a}{\sqrt{(\pi\nu t)}} + 2\right] \quad (a/\sqrt{(\nu t)} \gg 1). \tag{1}$$

This result, obtained by Howarth (1950) and Hasimoto (1951 a, b) (Davies (1950) also considers the flat strip, presenting a formal series solution which does not lend itself to analytic approximation for small t), omits any interaction between the strip edges, and it is such effects that we propose to calculate. It turns out that their magnitude is rather small, in the domain where the skin friction may be analysed by the preceding scheme, for the corrections to (1) are of exponential order; thus

$$D = \mu V\left[\frac{4a}{\sqrt{(\pi\nu t)}} + 2 - \frac{1}{2\pi}\left(\frac{\sqrt{(\nu t)}}{a}\right)^4 e^{-a^2/\nu t} + O\left\{\left(\frac{\sqrt{(\nu t)}}{a}\right)^6 e^{-a^2/\nu t}\right\}\right]$$
$$(a/\sqrt{(\nu t)} \gg 1). \tag{2}$$

The developments of the skin friction for the strip and the (smooth) circular cylinder show a quite different form after the first two terms, since the latter is, according to Batchelor (1952),

$$D = \mu V\left[\frac{2\pi a}{\sqrt{(\pi\nu t)}} + \pi - \frac{\sqrt{(\pi\nu t)}}{2a} + \frac{\pi\nu t}{4a^2} + \ldots\right] \quad (a/\sqrt{(\nu t)} \gg 1), \tag{3}$$

where a denotes the cylinder radius. For the later times, $\sqrt{(\nu t)}/a \gg 1$, Hasimoto (1954) and Batchelor have obtained other developments of the skin friction in the two cases; there is a smooth transition to the approximate expression (1) at $a/\sqrt{(\nu t)} = 1$. Hasimoto (1954, 1955) has recently generalized (3) so as to apply to a smooth general cylinder, but the corresponding accuracy is not yet attained for a polygonal cross-section.

The calculation to be described makes use of a double transform operation, one being a Laplace transform with respect to the time and the other a complex Fourier transform with respect to the (transverse) coordinate in the plane of the strip. After applying the Laplace transform to the differential equation for the fluid velocity, the resulting boundary

* An equivalent heat conduction problem relates to the variable temperature distribution in a uniform medium which is initially at zero temperature, and in which an internal boundary is maintained at a constant temperature. Here the heat flux at the boundary is analogous to the skin friction in the fluid motion problem.

value problem is formulated (§ 2) in terms of Green's functions and integral equations. The subsequent analysis is effectively carried out with complex Fourier transform and elementary function theoretic arguments, yielding finally a new integral equation (§ 3) which is well adapted to the determination of the skin friction (§ 4) in the initial phases of the motion. It is noteworthy that the inverse Laplace transform back to the time coordinate can be performed without approximation at any stage of iteration for the integral equation.

Similar Fourier transform techniques can be employed to analyse the scattering of sound or light waves incident on a strip, and the results invite comparison with those in the problem at hand. In the time-harmonic two-dimensional situation, with primary plane waves, the scattering cross-section at normal incidence relates to an integrated strip distribution and is thus analogous to the total skin friction. For comparison purposes, consider the skin friction (3) and the leading terms in the cross-section σ at wavelengths λ short compared with the strip width $2a$,

$$\sigma = 4a - \frac{4a}{\sqrt{\pi}}\left(\frac{\lambda}{2\pi a}\right)^{3/2}\cos\left(\frac{4\pi a}{\lambda} - \frac{\pi}{4}\right) + \frac{2a}{\pi}\left(\frac{\lambda}{2\pi a}\right)^{2}\cos\frac{8\pi a}{\lambda} \quad (\lambda/a \ll 1), \qquad (4)$$

when the total wave function vanishes at the strip. The counterpart of the Rayleigh term in the skin friction is the geometrical term of the cross-section, and whereas the next contribution to the skin friction arises from isolated edge effects, these do not contribute to the cross-section. Succeeding terms of the developments, which account for interaction between the strip edges, are of exponential form in the skin friction and of trigonometric form in the cross-section, reflecting the diffusion and propagation character of the problems.

Carrier (1955) has also studied the boundary layer aspect of solutions to certain types of integral equation, where reference is made to a semi-infinite configuration for the purpose of characterizing the edge behaviour. The solution to an inhomogeneous finite-range integral equation (cf. (21)) is given, neglecting interaction between the edges, and utilized in analysis of an oscillating plate in a viscous fluid.

2. Formulation

Let the strip be located in the (x, z)-plane,

$$-a \leqslant x \leqslant +a, \qquad -\infty < z < +\infty,$$

and set into motion at time $t = 0$ with steady velocity V along the z-direction. The viscous fluid begins a parallel motion, with velocity $v_z(x, y, t)$ which satisfies the equation

$$\frac{\partial v_z}{\partial t} = \nu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) \qquad (5)$$

(ν is the kinematic viscosity) and automatically complies with the equation of continuity, by virtue of translational invariance. The same differential equation applies to the velocity ratio

$$\psi(x, y, t) = v_z/V, \qquad (6)$$

which is further subject to the initial and boundary conditions

$$\psi = 0 \quad \text{at } t = 0, \tag{7}$$

$$\left.\begin{aligned} \psi &= 1 \quad \text{at } y = 0 \quad \text{and } |x| < a, \\ \psi &\to 0 \quad \text{as } x^2 + y^2 \to \infty, \end{aligned}\right\} \quad (t > 0) \tag{8}$$

and the symmetry condition

$$\psi(x, y, t) = \psi(x, -y, t). \tag{9}$$

If

$$\bar{\psi}(x, y, p) = p \int_0^\infty e^{-pt} \psi(x, y, t)\, dt \tag{10}$$

denotes a quantity proportional to the Laplace transform of $\psi(x, y, t)$, then, using (5), (6), (7), it follows that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right)\bar{\psi}(x, y, p) = 0 \tag{11}$$

and

$$\left.\begin{aligned} \bar{\psi} &= 1 \quad \text{at } y = 0 \quad \text{and } |x| < a, \\ \bar{\psi} &\to 0 \quad \text{as } x^2 + y^2 \to \infty, \end{aligned}\right\} \tag{12}$$

where

$$k^2 = p/\nu. \tag{13}$$

The boundary value problem stated in (11), (12), is advantageously reformulated with the help of solutions to the associated inhomogeneous differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right)G(x, y; x', y') = -\delta(x - x')\,\delta(y - y') \quad (\mathscr{R}\{k\} > 0). \tag{14}$$

A fundamental solution of this equation with logarithmic singularity at the 'source' point $x = x'$, $y = y'$, and a null value at infinity is the Green's function

$$G_0(x, y; x', y') = (2\pi)^{-1} K_0(k\sqrt{\{(x - x')^2 + (y - y')^2\}}), \tag{15}$$

where K_0 denotes the zero order Hankel function of imaginary argument, namely

$$K_0(\zeta) = \tfrac{1}{2}\pi i H_0^{(1)}(\zeta e^{i\pi/2}). \tag{16}$$

In terms of the source function G_0, a solution of the differential equation (11) which vanishes at infinity and is symmetric with respect to the plane of the strip can be represented by

$$\bar{\psi}(x, y, p) = \frac{1}{2\pi} \int_{-a}^{a} K_0(k\sqrt{\{(x - x')^2 + y^2\}}) g(x')\, dx'. \tag{17}$$

Placing $y = 0$, and writing

$$\bar{\psi}(x, 0, p) = f(x), \tag{18}$$

we have

$$f(x) = \frac{1}{2\pi} \int_{-a}^{a} K_0(k|x - x'|) g(x')\, dx', \tag{19}$$

and from the condition (12), which gives $f(x) = 1$ for $|x| < a$, there follows an integral equation for the (vorticity distribution function)

$$g(x) = -2\left(\frac{\partial\bar{\psi}}{\partial y}\right)_{y=0} \tag{20}$$

on the strip. The boundary value problem (11), (12), is thus linked with the solution of a one-dimensional integral equation,

$$1 = \frac{1}{2\pi}\int_{-a}^{a} K_0(k|x-x'|)g(x')\,dx' \quad (|x| < a), \tag{21}$$

and a knowledge of $g(x)$ determines $\bar{\psi}$ at all points off the strip by (17).

Choosing next the Green's function of the half space $y > 0$,

$$G_1(x,y;\,x',y') = (2\pi)^{-1}[K_0(k\sqrt{\{(x-x')^2+(y-y')^2\}}) - \\ - K_0(k\sqrt{\{(x-x')^2+(y+y')^2\}})], \tag{22}$$

which vanishes at $y = 0$, a representation for ψ therein is,

$$\bar{\psi}(x,y,p) = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_{|x'|<a} K_0(k\sqrt{\{(x-x')^2+y^2\}})\,dx' - \\ -\frac{1}{\pi}\frac{\partial}{\partial y}\int_{|x'|>a} K_0(k\sqrt{\{(x-x')^2+y^2\}})f(x')\,dx' \quad (y>0), \tag{23}$$

where the boundary value $\bar{\psi}(x,0,p) = f(x) = 1$ for $|x| < a$ is utilized. A further differentiation in (23) with respect to y, and passage to the limit $y = 0$, yields

$$-\tfrac{1}{2}g(x) = -\frac{1}{\pi}\frac{\partial^2}{\partial y^2}\left(\int_{|x'|<a} K_0(k\sqrt{\{(x-x')^2+y^2\}})\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k\sqrt{\{(x-x')^2+y^2\}})f(x')\,dx'\right)_{y\to 0+} \\ = \frac{1}{\pi}\left(\frac{d^2}{dx^2}-k^2\right)\left(\int_{|x'|<a} K_0(k|x-x'|)\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k|x-x'|)f(x')\,dx'\right), \tag{24}$$

for the K_0-function is a solution of the homogeneous equation (11) when $y' = 0$ and $y \to 0$. Since $\bar{\psi}(x,y,p) = \bar{\psi}(x,-y,p)$, it follows that $(\partial\bar{\psi}/\partial y)_{y=0} = 0$ for $|x| > a$, and the condition $g(x) = 0$ for $|x| > a$ provides an integro-differential equation,

$$\left(\frac{d^2}{dx^2}-k^2\right)\left(\int_{|x'|<a} K_0(k|x-x'|)\,dx' + \right. \\ \left. + \int_{|x'|>a} K_0(k|x-x'|)f(x')\,dx'\right) = 0 \quad (|x| > a), \tag{25}$$

for the transform $\bar{\psi}(x,0,p) = f(x)$, in the domain $|x| > a$. The solution to the boundary value problem (11), (12), can be based on this integro-differential equation, and the representation (23) for the even function of y, $\bar{\psi}(x,y,p)$, in the half-plane $y > 0$. It is evident that the two formulations given are of a complementary nature, with the primary role accorded to

the distribution $g(x)$ on the strip in the former, and to the coplanar distribution $f(x)$ off the strip in the latter; clearly, a knowledge of either function suffices to determine the other.

The skin friction $D(x)$ at the strip is

$$D(x) = -\mu\left(\frac{\partial v_z}{\partial y}\right)_{y=0} = \tfrac{1}{2}\mu V L^{-1}[g(x,p)], \tag{26}$$

where L^{-1} denotes an inverse Laplace transform, namely (recall the additional factor p introduced in (10)),

$$L^{-1}[g(x,p)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{g(x,p)}{p} e^{pt}\, dp \quad (c > 0). \tag{27}$$

In the total skin friction or drag, D, the integral of $D(x)$ over the strip width is involved, and an additional factor of 2 is necessary to account for both faces, whence

$$D = \mu V L^{-1}\left(\int_{-a}^{a} g(x,p)\, dx\right) = \frac{\mu V}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{dp}{p}\, e^{pt} \int_{-a}^{a} g(x,p)\, dx. \tag{28}$$

Although the drag is naturally related to $g(x)$, the latter function need not be the primary quantity in an integral equation formulation of the boundary value problem; to calculate D in the early stages of motion, it will in fact be convenient to work from the integro-differential equation for $f(x)$ and obtain $g(x)$ secondarily.

3. Integral equations

The basic integral equations (21) and (25) cannot be solved explicitly in closed form, and, as presently constituted, do not lend themselves to approximate solution, say, by iteration. Our objective is to obtain a new integral equation from (25) which is appropriate to the conditions prevailing at small values of t, and which also permits ready approximation to the skin friction. The idea underlying this transformation is that, for large values of the parameter $a/\sqrt{(\nu t)}$, the distributions $f(x)$ in the ranges $x > a$ and $x < -a$ are weakly coupled and exactly describable in the limit of no coupling, which corresponds to the geometrical configuration of a semi-infinite strip.

A brief exploration of the latter problem is in order, and information obtained relative to the behaviour of f and g near the edge will carry over to the finite strip case. The relation (19) provides a convenient starting point, and, if the origin is shifted to one edge, this relation becomes

$$\left.\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{2a} K_0(k|x-x'|)g(x')\, dx' \\ &= 1 \quad \text{for } 0 < x < 2a. \end{aligned}\right\} \tag{29}$$

Whence, letting $a \to \infty$, we obtain

$$\left.\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{\infty} K_0(k|x-x'|)g(x')\, dx' \\ &= 1 \quad \text{for } x > 0 \end{aligned}\right\} \tag{30}$$

as the form appropriate to the semi-infinite strip $0 < x < \infty$. Employing the integral representation

$$K_0(k\sqrt{\{(x-x')^2+(y-y')^2\}}) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\exp\{i\zeta(x-x')-\sqrt{(k^2+\zeta^2)}|y-y'|\}}{\sqrt{(k^2+\zeta^2)}}\,d\zeta, \tag{31}$$

and introducing

$$\bar{f}(\zeta) = \int_{-\infty}^{0} e^{-i\zeta x} f(x)\,dx, \tag{32}$$

$$\bar{g}(\zeta) = \int_{0}^{\infty} e^{-i\zeta x} g(x)\,dx, \tag{33}$$

we then obtain the transform version of (30):

$$\bar{f}(\zeta) + \frac{1}{\epsilon + i\zeta} = \frac{\bar{g}(\zeta)}{2\sqrt{(k^2+\zeta^2)}}. \tag{34}$$

Here ϵ appears by virtue of an exponential attenuation factor introduced when calculating the transform of $f = 1$ in the range $x > 0$. The relation (34) has meaning if there is a common domain of regularity in the complex ζ-plane for the functions occurring therein. According to (30), $f(x)$ contains the factor e^{kx} as $x \to -\infty$, and hence $\bar{f}(\zeta)$ is regular in the half-plane $\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}$; an exponential free behaviour for the strip distribution $g(x)$, as $x \to \infty$, implies that $\bar{g}(\zeta)$ is regular in the half-plane $\mathscr{I}\{\zeta\} < 0$. Thus, the common domain of regularity is a strip parallel to the real axis of the ζ-plane, defined by $0 > \mathscr{I}\{\zeta\} > -\mathscr{R}\{k\} = \mathscr{I}\{-ik\}$. Following the Wiener–Hopf procedure, (34) is rewritten in the form

$$\sqrt{(\zeta+ik)}\bar{f}(\zeta) + \frac{\sqrt{(\zeta+ik)}-\sqrt{(ik+i\epsilon)}}{\epsilon+i\zeta} = -\frac{\sqrt{(ik+i\epsilon)}}{\epsilon+i\zeta} + \frac{\bar{g}(\zeta)}{2\sqrt{(\zeta-ik)}}, \tag{35}$$

where the left-hand side is regular in the (upper) half-plane $\mathscr{I}\{\zeta\} > \mathscr{I}\{-ik\}$, and the right-hand side is regular in the (lower) half-plane $\mathscr{I}\{\zeta\} < 0$. An integral function is thus defined throughout the ζ-plane, and, since each side vanishes at infinity in the respective half-planes (assuming only that $f(x)$ and $g(x)$ are integrable at the edge $x = 0$), this function is zero. Hence,

$$\bar{f}(\zeta) = \frac{i}{\zeta}\left[1 - \sqrt{\left(\frac{ik}{\zeta+ik}\right)}\right], \tag{36}$$

in the limit $\epsilon = 0$, while

$$\bar{g}(\zeta) = 2\sqrt{(ik)}\,\frac{\sqrt{(\zeta-ik)}}{\epsilon+i\zeta}. \tag{37}$$

Inverting, we find

$$f(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\zeta x}}{\zeta}\left[1 - \sqrt{\left(\frac{ik}{\zeta+ik}\right)}\right] d\zeta$$

$$= \frac{i}{2\pi}\int_{-i\infty}^{i\infty} \frac{e^{kxu}}{u}\left[1 - \sqrt{\left(\frac{1}{1-u}\right)}\,du\right] = \frac{1}{\pi}\int_{1}^{\infty} \frac{e^{kxu}}{u\sqrt{(u-1)}}\,du \quad (x < 0), \tag{38}$$

and

$$g(x) = \frac{\sqrt{(ik)}}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\zeta x}}{\epsilon+i\zeta}\sqrt{(\zeta-ik)}\,d\zeta = -\frac{ik}{\pi}\int_{-i\infty}^{i\infty} \frac{e^{kxu}}{u+\epsilon/k}\sqrt{(u+1)}\,du$$

$$= 2k + \frac{2k}{\pi}\int_{1}^{\infty} \frac{e^{-kxu}}{u}\sqrt{(u-1)}\,du \quad (x > 0), \tag{39}$$

with k real. From (38), it appears that

$$\lim_{x\to 0-} f(x) = \frac{1}{\pi}\int_1^\infty \frac{du}{u\sqrt{(u-1)}} = 1, \tag{40}$$

which is the value of f for $x > 0$, and

$$f(x) \sim e^{kx}/(-kx)^{1/2} \quad \text{as } x \to -\infty. \tag{41}$$

According to (39),

$$g(x) \sim (kx)^{-1/2} \quad \text{as } x \to 0+ \tag{42}$$

and

$$g(x) \sim 2k \quad \text{as } x \to \infty. \tag{43}$$

Since

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_0(k|x-x'|)(2k)\,dx' \quad (-\infty < x < \infty), \tag{44}$$

the uniform distribution $g(x) = 2k$ evidently refers to a strip of infinite width. This synopsis of features of the semi-infinite strip configuration is sufficient for the present, and further references are deferred until later.

Turning attention again to the strip of finite width, and choosing (24) as a starting-point for the analysis, we find that the concomitant transform relation becomes

$$\tfrac{1}{2}\bar{g}(\zeta) = \sqrt{(k^2+\zeta^2)}\left[\bar{f}(\zeta) - \left(\frac{e^{i\zeta a}-e^{-i\zeta a}}{i\zeta}\right)\right], \tag{45}$$

where

$$\bar{g}(\zeta) = \int_{-a}^{a} e^{-i\zeta x} g(x)\,dx = \bar{g}(-\zeta), \tag{46}$$

and

$$\bar{f}(\zeta) = \bar{f}_1(\zeta) + \bar{f}_2(\zeta), \tag{47}$$

with

$$\bar{f}_1(\zeta) = \int_a^\infty e^{-i\zeta x} f(x)\,dx, \tag{48}$$

$$\bar{f}_2(\zeta) = \int_{-\infty}^{-a} e^{-i\zeta x} f(x)\,dx. \tag{49}$$

Evidently $\bar{g}(\zeta)$ is regular in the finite part of the ζ-plane, and with the local behaviour $g(x) \sim (k(a \mp x))^{-1/2}$ as $x \to \pm a$, which is characteristic of an isolated edge (see (42)), it follows that

$$\bar{g}(\zeta) \sim e^{\mp i\zeta a}/(\mp i\zeta)^{1/2} \quad \text{as } |\zeta| \to \infty \quad (\mathscr{I}\{\zeta\} \gtrless 0). \tag{50}$$

The exponential decrease of $f(x)$ as $x \to \infty$ implies that $\bar{f}_1(\zeta)$ is regular in the (lower) half-plane $\mathscr{I}\{\zeta\} < \mathscr{R}\{k\}$, and a finite limit for $f(x)$ as $x \to a+0$ (see (40)) yields the asymptotic behaviour

$$\bar{f}_1(\zeta) \sim e^{-i\zeta a}/\zeta \quad \text{as } |\zeta| \to \infty \qquad (\mathscr{I}\{\zeta\} < \mathscr{R}\{k\}) \tag{51}$$

in the half-plane of regularity. By symmetry, f is an even function of x, so that $\bar{f}_2(\zeta) = \bar{f}_1(-\zeta)$, and thus $\bar{f}_2(\zeta)$ is regular in the (upper) half-plane $\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}$, where

$$\bar{f}_2(\zeta) \sim e^{i\zeta a}/\zeta \quad \text{as } |\zeta| \to \infty \qquad (\mathscr{I}\{\zeta\} > -\mathscr{R}\{k\}). \tag{52}$$

Accordingly, the transform relation (45) holds in the strip $|\mathscr{I}\{\zeta\}| < \mathscr{R}\{k\}$ and, after the substitutions

$$\bar{f}_1(\zeta) = e^{-i\zeta a}\bar{F}_1(\zeta), \qquad \bar{f}_2(\zeta) = e^{i\zeta a}\bar{F}_2(\zeta), \tag{53}$$

takes the form

$$\tfrac{1}{2}\bar{g}(\zeta) = \sqrt{(k^2+\zeta^2)}[e^{-i\zeta a}\,\bar{F}_1(\zeta) + e^{i\zeta a}\,\bar{F}_2(\zeta) + (e^{i\zeta a} - e^{-i\zeta a})/i\zeta]. \tag{54}$$

Multiplying in turn by $e^{i\zeta a}$, $e^{-i\zeta a}$, we get

$$\tfrac{1}{2}\bar{g}(\zeta)e^{i\zeta a} = \sqrt{(k^2+\zeta^2)}[\bar{F}_1(\zeta) + e^{2i\zeta a}\,\bar{F}_2(\zeta) + (e^{2i\zeta a} - 1)/i\zeta], \tag{55}$$

$$\tfrac{1}{2}\bar{g}(\zeta)e^{-i\zeta a} = \sqrt{(k^2+\zeta^2)}[e^{-2i\zeta a}\,\bar{F}_1(\zeta) + \bar{F}_2(\zeta) + (1 - e^{-2i\zeta a})/i\zeta]. \tag{56}$$

On the factorization of the radical,

$$\sqrt{(k^2+\zeta^2)} = \sqrt{\{(\zeta+ik)(\zeta-ik)\}},$$

the first of these equations can be written

$$\tfrac{1}{2}\bar{g}(\zeta)\frac{e^{i\zeta a}}{\sqrt{(\zeta+ik)}} = \sqrt{(\zeta-ik)}\left[\bar{F}_1(\zeta) + e^{2i\zeta a}\,\bar{F}_2(\zeta) + \left(\frac{e^{2i\zeta a}-1}{i\zeta}\right)\right].$$

The left-hand member is regular in the upper half-plane, and the first term of the right-hand member is regular in the lower half-plane. For each remaining term of the right-hand member, a decomposition may be effected, with the help of Cauchy's integral theorem,

$$w(z) = \frac{1}{2\pi i}\oint \frac{w(\xi)}{\xi - z}\,d\xi,$$

into a sum of functions regular in the upper and lower half-planes, respectively. Collecting all terms regular in the respective overlapping

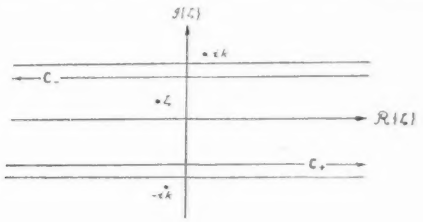

Figure 1.

half-planes, an integral function is defined throughout the ζ-plane, with the following representation in the lower half-plane:

$$\text{I.F.} = \sqrt{(\zeta-ik)}\,\bar{F}_1(\zeta) + \frac{1}{2\pi i}\int_{C_-} \sqrt{(\xi-ik)}e^{2i\xi a}\,\bar{F}_2(\xi)\,\frac{d\xi}{\xi-\zeta} + \\ + \frac{1}{2\pi i}\int_{C_-} \sqrt{(\xi-ik)}\left(\frac{e^{2i\xi a}-1}{i\xi}\right)\frac{d\xi}{\xi-\zeta}. \tag{57}$$

Here C_- is a path of integration conducted in the strip $|\mathcal{I}\{\zeta\}| < \mathcal{R}\{k\}$ (see figure 1), and ζ is located below the path. The terms in (57) tend to zero when $|\zeta| \to \infty$ in the lower half-plane, as do the corresponding terms at infinity in the upper half-plane, and thus the integral function vanishes, yielding

$$\sqrt{(\zeta-ik)}\,\bar{F}_1(\zeta) + \frac{1}{2\pi i}\int_{C_-} \sqrt{(\xi-ik)}e^{2i\xi a}\,\bar{F}_2(\xi)\,\frac{d\xi}{\xi-\zeta} + \\ + \frac{1}{2\pi i}\int_{C_-} \sqrt{(\xi-ik)}\left(\frac{e^{2i\xi a}-1}{i\xi}\right)\frac{d\xi}{\xi-\zeta} = 0. \tag{58}$$

Similar considerations applied to (56) give

$$\sqrt{(\zeta+ik)}\,\bar{F}_2(\zeta)+\frac{1}{2\pi i}\int_{C_+}\sqrt{(\xi+ik)}e^{-2i\xi a}\,\bar{F}_1(\xi)\frac{d\xi}{\xi-\zeta}+ \\ +\frac{1}{2\pi i}\int_{C_+}\sqrt{(\xi+ik)}\left(\frac{1-e^{-2i\xi a}}{i\xi}\right)\frac{d\xi}{\xi-\zeta}=0, \quad (59)$$

where C_+ is another path conducted in the strip, this time below the point ζ.

The integrals on C_-, C_+ are usefully recast by deforming the contours in the upper and lower half-planes, respectively, so as to run along the branch cuts drawn from $\zeta=\pm ik$. If these cuts are placed on the imaginary axis away from the origin ($\mathscr{I}\{k\}\to 0$), then, taking account of the phases of the radicals at each side of a cut, it follows that

$$\sqrt{(\zeta-ik)}\,\bar{F}_1(\zeta)=-\frac{e^{i\pi/4}}{\pi}k^{1/2}\int_1^\infty\sqrt{(u-1)}\left(\frac{e^{-2kau}-1}{u}\right)\frac{du}{iku-\zeta}+ \\ +\frac{e^{i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\,\bar{F}_2(iku)e^{-2kau}\frac{du}{iku-\zeta} \\ =\frac{e^{i\pi/4}}{\pi}k^{1/2}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{iku-\zeta}+ \\ +\frac{e^{i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{iku-\zeta}, \quad (60)$$

and

$$\sqrt{(\zeta+ik)}\,\bar{F}_2(\zeta)=\frac{e^{3i\pi/4}}{\pi}k^{1/2}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{iku+\zeta}+ \\ +\frac{e^{3i\pi/4}}{\pi}k^{3/2}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_1(-iku)-\frac{1}{ku}\right)e^{-2ku}\frac{du}{iku+\zeta}. \quad (61)$$

The integral equations (60), (61) are not independent, for reversal of the sign of ζ in (60), and the use of relation $\bar{F}_1(\zeta)=\bar{F}_2(-\zeta)$, leads to (61). Let $\zeta=ikv$ in (61); then

$$\sqrt{(v+1)}\,\bar{F}_2(ikv)=\frac{1}{\pi k}\int_1^\infty\frac{\sqrt{(u-1)}}{u}\frac{du}{u+v}+ \\ +\frac{1}{\pi}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{u+v} \\ =-\frac{1}{kv}+\frac{\sqrt{(v+1)}}{kv}+ \\ +\frac{1}{\pi}\int_1^\infty\sqrt{(u-1)}\left(\bar{F}_2(iku)-\frac{1}{ku}\right)e^{-2kau}\frac{du}{u+v}, \quad (62)$$

which is an integral equation for the function $\bar{F}_2(ikv)$. When $a\to\infty$, the integral drops out, and the resulting form of $\bar{F}_2$ agrees with that found in the semi-infinite strip configuration, namely (36). A more symmetrical version of the integral equation is obtained by defining

$$P(v)=\frac{1}{v}-k\,\bar{F}_2(ikv), \quad (63)$$

for then

$$\sqrt{(v+1)}P(v) = \frac{1}{v} + \frac{1}{\pi}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u+v}. \tag{64}$$

From (62) it also follows that

$$\bar{F}_1(0) = \bar{F}_2(0) = \frac{1}{2k} - \frac{1}{\pi k}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u}. \tag{65}$$

The relations (64), (65) provide a suitable basis for calculating the skin friction in the early stages of the motion, as will now be verified.

4. Skin friction

To begin the skin friction calculation, an expression for $\bar{g}(0)$ is required. Using (54), (65), this is given by

$$\bar{g}(0) = 4ka + 4k\,\bar{F}_1(0) = 4ka + 2 - \frac{4}{\pi}\int_1^\infty \sqrt{(u-1)}P(u)e^{-2kau}\frac{du}{u}, \tag{66}$$

where $P(u)$ satisfies the integral equation (64). An iterative procedure can be applied to the integral equation when $\mathscr{R}\{k\} > 0$, and the leading terms of $P(u)$ thus obtained are

$$P(u) = \frac{1}{u\sqrt{(u+1)}} + \frac{1}{\pi\sqrt{(u+1)}}\int_1^\infty \sqrt{\left(\frac{v-1}{v+1}\right)}\frac{e^{-2kav}}{v}\frac{dv}{u+v}. \tag{67}$$

The corresponding approximation for $\bar{g}(0,p)$ is

$$\bar{g}(0,p) = 4a\sqrt{\left(\frac{p}{\nu}\right)} + 2 - \frac{4}{\pi}\int_1^\infty \sqrt{\left(\frac{u-1}{u+1}\right)}e^{-2a+\sqrt{(p/\nu)}u}\frac{du}{u^2} - \\ - \frac{4}{\pi^2}\int_1^\infty \sqrt{\left(\frac{(u-1)(v-1)}{(u+1)(v+1)}\right)}e^{-2a\sqrt{(p/\nu)}(u+v)}\frac{dudv}{uv(u+v)}, \tag{68}$$

after reverting to the Laplace transform variable,

$$p = \nu k^2. \tag{69}$$

There remains only an evaluation of the inversion integral (28),

$$D = \frac{\mu V}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\bar{g}(0,p)\frac{dp}{p} \quad (c > 0), \tag{70}$$

and it will be noted that the conditions $\mathscr{R}\{p\} > 0$, $\mathscr{R}\{k\} > 0$ are indeed compatible with (69). The resulting skin friction is

$$D = 2\mu V\left[\frac{2a}{\sqrt{(\pi\nu t)}} + 1 - \frac{2}{\pi}\int_1^\infty \sqrt{\left(\frac{u-1}{u+1}\right)}\operatorname{erfc}\left(\frac{au}{\sqrt{(\nu t)}}\right)\frac{du}{u^2} - \right. \\ \left. - \frac{2}{\pi^2}\int_1^\infty \sqrt{\left(\frac{(u-1)(v-1)}{(u+1)(v+1)}\right)}\operatorname{erfc}\left(\frac{a(u+v)}{\sqrt{(\nu t)}}\right)\frac{dudv}{uv(u+v)}\right], \tag{71}$$

where

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\xi^2}\,d\xi \tag{72}$$
$$\sim e^{-z^2}/\sqrt{\pi}z \quad \text{for } z \gg 1$$

is the complementary error function.

When $a/\sqrt{(\nu t)} \gg 1$, the principal contributions to the integrals arise from the neighbourhood of the lower limit, and thus the double integral is smaller than the single integral by a factor $e^{-3a^2/\nu t}$. This exponential decrease in magnitude evidently characterizes further contributions to the skin friction resulting from continued iteration of the integral equation. Successive terms acquire comparable magnitude when $a/\sqrt{(\nu t)} \ll 1$, and, in the limit $a/\sqrt{(\nu t)} \to 0$, the full iterated series is required to compensate the time independent term in D. The development (71) is therefore useful only for the early stages of the motion, as anticipated.

The first term in (71) arises from a uniform, local skin friction characteristic of an infinite strip, and the next term takes account of the non-uniformity introduced by the strip edges, here computed additively when the latter are infinitely remote. To verify this interpretation, observe that the excess skin friction on the semi-infinite strip $0 < x < \infty$, attributable to the edge, stems from the distribution (see (39))

$$g(x) = \frac{2k}{\pi} \int_1^\infty \sqrt{(u-1)} e^{-kxu} \frac{du}{u} \quad (x > 0), \tag{73}$$

whose integrated value over the range $0 < x < \infty$ is unity. The second term in (68), and hence (71), thus bespeaks two such contributions, from remote edges. If the distribution (73) is integrated over a strip $0 < x < 2a$, there is an additional contribution which is evidently distinct from the third term of (68). The necessary refinement of $g(x)$ entails a direct consideration of the finite strip configuration, and it is of interest to see how this may be carried out in the transform formulation.

In $|x| < a$, by (24),

$$-\tfrac{1}{2} g(x) = \frac{1}{2\pi}\left(\frac{d^2}{dx^2} - k^2\right)\left[\int_{-\infty}^{\infty} e^{i\zeta x} \frac{2 \sin \zeta a}{\zeta} \frac{d\zeta}{\sqrt{(k^2+\zeta^2)}} + \right.$$
$$\left. + \int_{-\infty}^{\infty} e^{i\zeta x}\{e^{-i\zeta a} \bar{F}_1(\zeta) + e^{i\zeta a} \bar{F}_2(\zeta)\} \frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}\right], \tag{74}$$

$$\bar{F}_1(\zeta) = -\frac{i}{\zeta}\left\{1 - \sqrt{\left(\frac{-ik}{\zeta - ik}\right)}\right\} -$$
$$- \frac{e^{i\pi/2}}{\pi} \sqrt{\left(\frac{-ik}{\zeta - ik}\right)} \int_1^\infty \sqrt{(u-1)} P(u) e^{-2kau} \frac{du}{iku - \zeta} = \bar{F}_2(-\zeta). \tag{75}$$

A modification of the integrals is expedient, and results from deforming the contour into the upper or lower part of the ζ-plane. Thus, for $|x| < a$,

$$\int_{-\infty}^{\infty} e^{i\zeta x} \frac{2 \sin \zeta a}{\zeta} \frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}$$
$$= \frac{1}{2i} \int_{-\infty}^{\infty} \{(1 - e^{-i\zeta(a+x)}) - (1 - e^{i\zeta(a+x)}) +$$
$$+ (1 - e^{-i\zeta(a-x)}) - (1 - e^{i\zeta(a-x)})\} \frac{d\zeta}{\zeta\sqrt{(k^2+\zeta^2)}}$$
$$= \frac{2}{k} \int_1^\infty (2 - e^{-k(a+x)u} - e^{-k(a-x)u}) \frac{du}{u\sqrt{(u^2-1)}}, \tag{76}$$

and

$$\int_{-\infty}^{\infty} e^{-i\zeta(a-x)}\bar{F}_1(\zeta)\frac{d\zeta}{\sqrt{(k^2+\zeta^2)}}$$
$$= \frac{2}{k}\int_1^{\infty}\frac{e^{-k(a-x)u}}{u}\left(1-\frac{1}{\sqrt{(u+1)}}\right)\frac{du}{\sqrt{(u^2-1)}} -$$
$$-\frac{2}{\pi k}\int_1^{\infty}\frac{e^{-k(a-x)u}}{\sqrt{(u+1)}}\left\{\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{\sqrt{(u^2-1)}}, \quad (77)$$

while the corresponding integral with $\bar{F}_2(\zeta)$ calls only for a reversal in the sign of x in the above expressions. Substituting these expressions in (74), we obtain

$$g(x) = 2k + \frac{2}{\pi k}\left(\frac{d^2}{dx^2}-k^2\right)\left[\int_1^{\infty}\frac{e^{-k(a+x)u}+e^{-k(a-x)u}}{\sqrt{(u+1)}}\times\right.$$
$$\left.\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{\sqrt{(u^2-1)}}\right]$$
$$= 2k + \frac{2k}{\pi}\left[\int_1^{\infty}\sqrt{(u-1)}(e^{-k(a+x)u}+e^{-k(a-x)u})\times\right.$$
$$\left.\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}du\right]$$
$$= 2k + \frac{2k}{\pi}\int_1^{\infty}\sqrt{(u^2-1)}(e^{-k(a+x)u}+e^{-k(a-x)u})P(u)\,du, \quad (78)$$

where the last form follows from a use of the integral equation for $P(u)$. In the second form, there may be recognized contributions of the type (73), referred to each strip edge. The singularities of $g(x)$ at $x = \pm a$ are contained in factors $(a \pm x)^{-1/2}$, which are also characteristic of an isolated edge. The time varying function $g(x, t)$ obtains after the inverse Laplace transform is applied to (78), and, with the first approximation $P(u) = 1/u\sqrt{(u+1)}$, the result is

$$g(x,t) = \frac{2}{\sqrt{(\pi\nu t)}} + \frac{2}{\pi\sqrt{(\pi\nu t)}}\int_1^{\infty}\sqrt{(u-1)}\times$$
$$\times\left\{\exp\left(-\frac{(a+x)^2u^2}{4\nu t}\right)+\exp\left(-\frac{(a-x)^2u^2}{4\nu t}\right)\right\}\frac{du}{u}. \quad (79)$$

After integration of (78), one finds

$$\int_{-a}^{a} g(x)\,dx = 4ka + \frac{4}{\pi}\int_1^{\infty}\sqrt{(u-1)}(1-e^{-2kau})\times$$
$$\times\left\{\frac{1}{u}+\frac{1}{\pi}\int_1^{\infty}\sqrt{(v-1)}P(v)e^{-2kav}\frac{dv}{u+v}\right\}\frac{du}{u} \quad (80\,a)$$
$$= 4ka + \frac{4}{\pi}\int_1^{\infty}\sqrt{(u^2-1)}(1-e^{-2kau})P(u)\frac{du}{u}, \quad (80\,b)$$

in contrast with (66). However, on substitution of the iterative expression for $P(u)$, the same development is obtained with each of these representations; it is noteworthy that (66) and (80 a) lead to the result (68) when $P(u)$ is given by the two-term approximation (67), whereas a three-term approximation

is called for in (80 b). This feature reflects a difference in basis of the calculation according as the off-strip distribution $f(x)$ for $|x| > a$, or the strip distribution $g(x)$ for $|x| < a$, is involved. The semi-infinite distribution is a natural first approximation for $f(x)$, and each successive stage of approximation generates a distinct order-of-magnitude correction in both $f(x)$ and the skin friction D. On the other hand, only a finite part of the semi-infinite distribution for $g(x)$ is relevant, and successive approximations are partially coupled in magnitude, resulting in like order contributions to D.

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The asymptotic expansions at large Reynolds numbers for steady motion between non-coaxial rotating cylinders

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SUMMARY

The motion of a viscous fluid contained between two rotating, circular cylinders whose axes are set slightly apart is considered. The equations of viscous motion are linearized by expanding the stream function in the form $\sum_{n=0}^{\infty} \psi_n \gamma^n$, where γ is a parameter which depends on the distance between the cylinder axes. The ensuing analysis appears to hold for all values of the fluid viscosity ν, and in particular for small values of ν.

The asymptotic behaviour of the solutions for small ν is examined, attention being mainly confined to the first order stream function ψ_1 and the corresponding component of vorticity ζ_1. Outside the boundary layers, where, for small ν, ψ_1 may be expanded asymptotically as $\sum_{n=0}^{\infty} \psi_1^{(n)} \nu^{n/2}$, the terms $\zeta_1^{(n)}$ of the corresponding expansions for the vorticity are shown to be uniform throughout the fluid. It is noted that the asymptotic expansions of ψ_1 for the region of the boundary layers and for the region outside the boundary layers may be combined in a single expansion which holds in both regions. The leading terms of this expansion are calculated by boundary layer methods.

1. INTRODUCTION

This paper is concerned with a viscous motion whose governing equations may, in principle, be solved exactly. The asymptotic behaviour of the solution when the fluid viscosity ν is small is derived and discussed and a procedure formulated by which the higher order corrections to the boundary layer approximation may be determined. Inasmuch as the streamlines of the motion are closed the results bear particularly on closed flows.

Now, the boundary layer approximation, as conceived by Prandtl, determines the stream function ψ of a steady two-dimensional motion with an error which is, in general, $O(UL\{\nu/UL\})$ in the boundary layer and $O(UL\{\nu/UL\}^{1/2})$ elsewhere, U and L being respectively a typical speed and a typical length of the motion. For certain configurations even this accuracy cannot be attained. If for example the fluid impinges on a sharp edge as in the uniform streaming past a flat plate, the Prandtl approximation

fails to determine the stream function correctly near the edge. More serious difficulties attend the shedding of a boundary layer to form a wake. Such configurations aside, the problem remains of how best to determine the higher order corrections to the original approximation.

One correction can be made at once; that is, the displacement of the streamlines outside the boundary layer caused by the slow movement of fluid inside the layer can be calculated. In a recent paper, Kaplun (1954) has shown that by suitable choice of coordinates the solution of the boundary layer approximation can be made to hold throughout the whole region of flow and made to include, outside the boundary layer, the effects of displacement thickness.

In order to formulate a procedure for determining the higher order corrections, it is helpful to have as a guide one or more exact solutions to the equations of motion, whose asymptotic behaviour for small ν has been elucidated. One such solution, though for an unsteady motion, has been analysed by Lagerstrom & Cole (1955). Lagerstrom & Cole consider the motion set up by a uniformly expanding circular cylinder moving parallel to its axis in unbounded fluid. The exact solutions obtained by Lagerstrom & Cole are of the form

$$\psi \sim \sum_{n=0}^{m} \mathscr{R}_n(\mathbf{r}, t, \nu)\nu^{-n/2} + o(\nu^{-m/2}),$$

where the coefficients $\mathscr{R}_n$ are significant in the boundary layer and elsewhere are transcendentally small, and they propose, from general considerations (which are not stated), that the stream function of steady motion with no wake may be generally represented in the form

$$\psi \sim \sum_{n=0}^{m} \mathscr{I}_n(\mathbf{r})\nu^{-n/2} + \sum_{n=0}^{m} \mathscr{R}_n(\mathbf{r}, \nu)\nu^{-n/2} + o(\nu^{-m/2}),$$

where the first term represents the truncated asymptotic series for ψ at a fixed point $\mathbf{r}$.

The example considered in the following concerns the motion of fluid between two rotating, circular cylinders whose axes are parallel and set slightly apart. When the separation of the cylinder axes is small compared with the difference in length of their radii the equations of motion may be solved by a perturbation method. Moreover the perturbation expansions appear to hold uniformly for all ν, providing that the expansion parameter is sufficiently small. The solution therefore determines the asymptotic behaviour of the motion when ν is small.

2. Linearization of the equations of motion

We begin by defining the perturbation expansions and deriving equations of motion for the first order terms.

Let the radii of the two cylinders be a, b and the distance between their axes $a\epsilon$. Take polar coordinates r, θ in any cross-section, with origin at the axis of the inner cylinder and reference line $\theta = 0$ along the radius to the axis of the outer cylinder (see figure 1). Introduce further the non-dimensional

coordinates ρ, ϕ defined in terms of r, θ by

$$\left.\begin{aligned} z &= a\,\frac{\zeta+\gamma}{1+\gamma\zeta}, \\ z &= re^{i\theta}, \qquad \zeta = \rho e^{i\phi}, \end{aligned}\right\} \tag{1}$$

with

$$\gamma = -2\epsilon[(b/a)^2-1-\epsilon^2+\sqrt{\{(b^2/a^2-1-\epsilon^2)^2-4\epsilon^2\}}]^{-1}. \tag{2}$$

The circular sections of the inner and outer cylinders are then the coordinate lines $\rho = 1$ and $\rho = \beta$, where

$$\beta = \frac{(b/a)+\epsilon-\gamma}{1-(b/a)\gamma-\epsilon\gamma}. \tag{3}$$

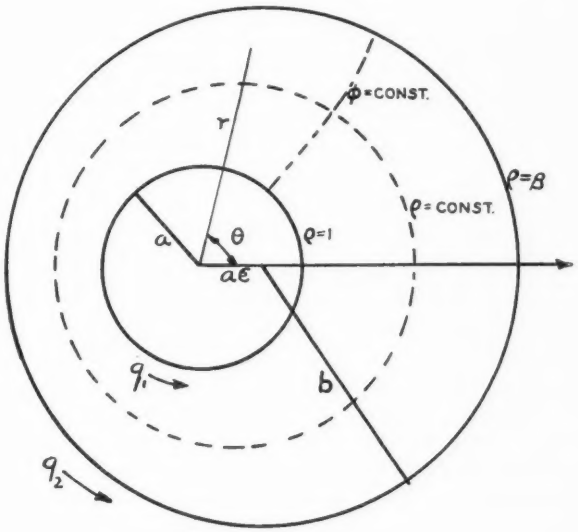

Figure 1.

The remaining coordinate lines ρ = constant and ϕ = constant constitute the two orthogonal pencils of circles generated by the circular cylinder sections The coordinate mesh is clearly the same as that of the two-dimensional bipolar system in which the cylinder sections are coordinate lines. The coordinate variables ρ, ϕ are however different, having been modified so as to have the advantage in this problem of degenerating to polar variables r, θ when the cylinders are coaxial ($\epsilon = \gamma = 0$).

The rotary motion between cylinders is supposed two-dimensional and may therefore be represented in terms of a non-dimensional stream function ψ, the velocity components in the coordinate directions ρ increasing, ϕ increasing, being respectively

$$u_\rho = q_1\frac{\sqrt{J}}{\rho}\frac{\partial\psi}{\partial\phi}, \qquad u_\phi = -q_1\sqrt{J}\frac{\partial\psi}{\partial\rho}, \tag{4}$$

where q_1 is the peripheral speed of the inner cylinder. The function J is in effect the Jacobian of the transformation (1) and is defined by

$$J = \frac{(1+2\gamma\rho\cos\phi+\gamma^2\rho^2)^2}{(1-\gamma^2)^2}. \tag{5}$$

On eliminating the pressure, the equations of motion reduce to

$$-\frac{1}{\rho}\frac{\partial(\psi,\zeta)}{\partial(\rho,\phi)} = \frac{1}{R}\nabla^2\zeta, \qquad \zeta = -J\nabla^2\psi, \tag{6}$$

where ζ is the axial component of vorticity, R is the Reynolds number, defined here by aq_1/ν, and, in both equations,

$$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}. \tag{7}$$

(In using q_1 as a typical speed, it has been tacitly assumed that the inner cylinder is not at rest. If it were at rest, the typical speed could with only trivial changes be taken to be the peripheral speed q_2 of the outer cylinder.) The boundary conditions at the inner and outer cylinders are respectively,

$$\left.\begin{aligned} &\psi(1,\phi) = 0, && \frac{\partial}{\partial\rho}\psi(1,\phi) = -\frac{1}{\sqrt{\{J(1,\phi)\}}}, \\ &\psi(\beta,\phi) = \text{constant}, && \frac{\partial}{\partial\rho}\psi(\beta,\phi) = -\frac{q_2}{q_1\sqrt{\{J(\beta,\phi)\}}}. \end{aligned}\right\} \tag{8}$$

Further the solution to (6) must be such that the pressure is single-valued. These equations and boundary conditions suffice to determine ψ. From them it is clear that ψ depends on the geometrical parameters β, γ, the ratio of the peripheral speeds q_1/q_2, and the Reynolds number R.

It is now proposed to consider the motion when the separation $a\epsilon$ of the cylinder axes is small compared with the difference in length $b-a$ of their radii (that is, $\gamma \ll 1$). In the particular instance where the cylinders are coaxial the motion is axi-symmetric and independent of the Reynolds number. The stream function is then

$$\psi_0(\rho,\beta,q_2/q_1) = -\tfrac{1}{2}A\rho^2 - B\log\rho, \tag{9}$$

where

$$A = \frac{\beta(q_2/q_1)-1}{\beta^2-1}, \qquad B = \frac{\beta^2-\beta(q_2/q_1)}{\beta^2-1}; \tag{10}$$

and the vorticity is

$$\zeta_0(\beta,q_2/q_1) = 2A. \tag{11}$$

In general, when the cylinders are nearly coaxial, it is assumed that the stream function ψ and the vorticity component ζ may be expanded in the forms

$$\left.\begin{aligned} \psi &= \psi_0(\rho,\beta,q_2/q_1) + \sum_{n=1}^{\infty}\psi_n(\rho,\phi,\beta,q_2/q_1,R)\gamma^n, \\ (1-\gamma^2)^2\zeta &= 2A(\beta,q_2/q_1) + \sum_{n=1}^{\infty}\zeta_n(\rho,\phi,\beta,q_2/q_1,R)\gamma^n. \end{aligned}\right\} \tag{12}$$

It is the first order motion described by the first order terms with which this paper is principally concerned. The higher order terms are discussed briefly in the final section.

Concerning the mode of expansion, note that the coordinate description of the bounding surfaces does not involve the expansion parameter γ. This is important if the expansion is to be used for large Reynolds numbers, and is the reason for introducing 'bipolar' coordinates. If the coordinate description of the bounding surfaces involved γ then the boundary conditions would have to be applied at surfaces $\gamma = 0$ different from the cylinders. For this purpose, the velocity field at each boundary would have to be expanded in powers of 'distance' from the surface $\gamma = 0$ to the boundary, and it is unlikely that such series would converge when R is large. Note also that when $\gamma \neq 0$, the motion described by $\psi_0(\rho, q_2/q_1, \beta)$ is not axi-symmetric. In this motion the circles $\rho =$ constant and, in particular, the cylinders are streamlines, the speeds at the inner and outer cylinders being respectively $q_1\sqrt{\{J(1, \phi, \gamma)\}}$, $q_2\sqrt{\{J(\beta, \phi, \gamma)\}}$.

When the expansions of ψ, ζ and the analogous expansion of J are substituted into the equations of motion (6) and the boundary conditions (8), and the leading terms isolated, we get

$$\left(A + \frac{B}{\rho^2}\right)\frac{\partial \zeta_1}{\partial \phi} = \frac{1}{R}\nabla^2\zeta_1, \tag{13 a}$$

$$\zeta_1 = -\nabla^2(\psi_1 - A\rho^3\cos\phi), \tag{13 b}$$

with the boundary conditions on ψ_1:

$$\left.\begin{aligned} &\psi_1(1, \phi) = 0, \qquad \frac{\partial}{\partial\rho}\psi_1(1, \phi) = 2\cos\phi, \\ &\psi_1(\beta, \phi) = \text{constant}, \quad \frac{\partial}{\partial\rho}\psi_1(\beta, \phi) = 2\,q_2/q_1\,\beta\cos\phi. \end{aligned}\right\} \tag{14}$$

Further, from the corresponding exact condition, the solution of (13) must be such that the first order perturbation in pressure is single-valued.

The equations governing the first order motion contain the Reynolds number in the same way as the exact equations but have the merit of being linear and hence more tractable. In that the convective velocity in the vorticity equation (13) is known, this equation is mathematically similar to Oseen's equation and the generalisation of it discussed by Zeilon (1927) and Burgers (1921). In the present case, there is, however, reason to believe that the perturbation to ψ described by the equations (13) is uniformly valid to $O(\gamma)$ even at large Reynolds numbers. A similar linearization of the equations of motion was discussed by Proudman (1956) who investigated the flow between two concentric spheres which rotate with slightly different angular velocities. In the simpler configuration considered here the linearized equations may be solved exactly. This we proceed to do.

3. The first order solution

It is evident from (13) and (14), that ψ_1 and ζ_1 vary with ϕ according to the simple forms

$$\left.\begin{aligned} &\psi_1 = A\rho^3\cos\phi + 2\mathscr{R}\{f(\rho)e^{i\phi}\}, \\ &\zeta_1 = 2\mathscr{R}\{g(\rho)e^{i\phi}\}. \end{aligned}\right\} \tag{15}$$

The way in which ψ_1 and ζ_1 vary with ρ is then determined from the differential equations

$$\left.\begin{aligned} g'' + \frac{g'}{\rho} - \left(RAi + \frac{RBi+1}{\rho^2}\right)g = 0, \\ f'' + \frac{f'}{\rho} - \frac{f}{\rho^2} = -g, \end{aligned}\right\} \tag{16}$$

with the boundary conditions

$$\left.\begin{aligned} f(1) = -\tfrac{1}{2}A, \qquad & f'(1) = 1 - \tfrac{3}{2}A, \\ f(\beta) = -\tfrac{1}{2}A\beta^2, \qquad & f'(\beta) = \frac{q_2}{q_1}\beta - \tfrac{3}{2}A\beta^2, \end{aligned}\right\} \tag{17}$$

where primes denote differentiation with respect to ρ.

Provided that the vorticity of the motion described by ψ_0 is not zero ($A \neq 0$), $g(\rho)$ satisfies a Bessel's equation of order $\mu = \sqrt{(BRi+1)}$ with independent variable $z = \sqrt{(-ARi)}\rho$. Thus

$$g(\rho) = -2CJ_\mu(z) - 2DH^1_\mu(z), \tag{18}$$

where $J_\mu(z)$, $H^1_\mu(z)$ denote respectively the Bessel function and Hankel function of order μ, and C, D are constants to be determined. For definiteness we take $|\arg\mu| < \frac{1}{2}\pi$ and $|\arg z| = \frac{1}{4}\pi$. A general integral of (16) may now be seen to be

$$f(\rho) = A_1\rho + \frac{B_1}{\rho} + CI_1(\rho) + DI_2(\rho), \tag{19}$$

where

$$\left.\begin{aligned} I_1(\rho) &= \rho\int_1^\rho J_\mu(z)\,d\rho - \frac{1}{\rho}\int_1^\rho \rho^2 J_\mu(z)\,d\rho, \\ I_2(\rho) &= \rho\int_\beta^\rho H^1_\mu(z)\,d\rho - \frac{1}{\rho}\int_\beta^\rho \rho^2 H^1_\mu(z)\,d\rho. \end{aligned}\right\} \tag{20}$$

The constants A_1, B_1, C, D may be evaluated from the boundary conditions (17). Whence

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2} - A - \tfrac{1}{2}D\{I_2(1) + I'_2(1)\}, \\ B_1 &= -\tfrac{1}{2}B - \tfrac{1}{2}D\{I_2(1) - I'_2(1)\}, \\ \Delta C &= \left(\beta + \frac{1}{\beta} - \frac{2q_2}{q_1}\right)I_2(1) + \left(\beta - \frac{1}{\beta}\right)I'_2(1), \\ \Delta D &= \left(2 - \beta - \frac{1}{\beta}\right)I_1(\beta) + (\beta^2 - 1)\frac{q_2}{q_1}I'_2(\beta), \end{aligned}\right\} \tag{21}$$

where

$$\Delta = \left(1 - \frac{1}{\beta^2}\right)\{\beta I'_1(\rho)I'_2(1) - I_1(\beta)I_2(1)\} + \\ + \left(1 + \frac{1}{\beta^2}\right)\{\beta I'_1(\beta)I_2(1) - I_1(\beta)I'_2(1)\}. \tag{22}$$

Thus, when the vorticity of the unperturbed motion is non-zero ($A \neq 0$), the solution for ψ_1 is

$$\psi_1 = A\rho^3 \cos\phi + 2\mathscr{R}\left\{\left[A_1\rho + \frac{B_1}{\rho} + CI_1(\rho) + DI_2(\rho)\right]e^{i\phi}\right\}, \tag{23}$$

where I_1, I_2, A_1, B_1, C, D are defined in (20) to (22); whilst the solution for ζ_1 is

$$\zeta_1 = -4\mathscr{R}\{[CJ_\mu(z) + DH^1_\mu(z)]e^{i\mu\phi}\}. \tag{24}$$

The exceptional case in which the motion described by ψ_0 is irrotational ($A = 0$) arises when $q_2/q_1 = 1/\beta$. In this case $g(\rho)$ has the form

$$g(\rho) = -2C\rho^\mu - 2D\rho^{-\mu}. \tag{25}$$

The function $f(\rho)$ may be obtained in the same way as when $A \neq 0$, the 'Bessel' functions $J_\mu(z)$, $H^1_\mu(z)$ being replaced by ρ^μ, $\rho^{-\mu}$. Thus

$$f(\rho) = A_1\rho + \frac{B_1}{\rho} + CI_1(\rho) + DI_2(\rho), \tag{26}$$

where I_1, I_2 are now

$$\left.\begin{aligned} I_1(\rho) &= \frac{\rho}{1+\mu}(\rho^{1+\mu} - 1) - \frac{1}{(3+\mu)\rho}(\rho^{3+\mu} - 1), \\ I_2(\rho) &= \frac{\rho}{1-\mu}(\rho^{1-\mu} - \beta^{1-\mu}) - \frac{1}{(3-\mu)\rho}(\rho^{3-\mu} - \beta^{3-\mu}). \end{aligned}\right\} \tag{27}$$

The constants A_1, B_1, C, D are again related to I_1 and I_2 as in (21). The solution for ψ_1 is therefore formally the same as in (23), the leading term now vanishing and the functions $I_1(\rho)$, $I_2(\rho)$ being defined as in (27). The solution for ζ_1, however, changes to

$$\zeta_1 = -4\mathscr{R}\{[C\rho^\mu + D\rho^{-\mu}]e^{i\beta}\}. \tag{28}$$

The significance of these solutions is that they express in closed form the dependence of a viscous motion on Reynolds number. In what follows they are used to derive the asymptotic behaviour of the motion when the Reynolds number is large. Certain properties of this motion may of course be anticipated. Thus we may expect viscous action to be negligible save in thin layers at the cylinders, and in these layers the Prandtl approximation should hold. Further, the vorticity at infinite Reynolds number should be uniform, save at the cylinders (Batchelor 1956). Our main interest in deriving the asymptotic behaviour of ψ_1, however, lies in the higher order corrections to the Prandtl approximation, or, equally, in the higher order terms of the asymptotic representations of ψ_1.

4. Behaviour of the solution to the first order vorticity equation for large R

As a preliminary to deriving the behaviour of the first order stream function ψ_1 for large R, it is helpful to consider first the behaviour of the vorticity component ζ_1. It is sufficient, for this purpose, to confine attention to those properties that follow directly from the vorticity equation (13 a), when the way in which the vorticity component ζ_1 varies periodically with ϕ is known. In view of the similarity of the approximate vorticity equation to the exact vorticity equation expressed in coordinates for which the streamlines are coordinate lines, these properties may be of interest in suggesting general properties of the vorticity distribution for motions with closed streamlines.

Before proceeding further, it should be noted that the method of expanding the stream function as a series in γ cannot be expected to hold for large R if the outer cylinder is at rest. The reason for this may be seen by referring to the boundary layer approximation. If the series were valid and a boundary layer existed on the outer cylinder, the speed of the inviscid flow at the outer cylinder would be expected to be $O(\gamma q_1)$. The thickness of the boundary layer there would then be $O(a\gamma^{-1/2}R^{-1/2})$. Thus for a given value of γ and a Reynolds number much greater than $1/\gamma$ (a necessary condition for the boundary layer to exist), the flow quantities of the boundary layer would not be analytic functions of γ. For instance, the velocity gradient $\partial u_\phi/\partial\rho$ at the outer cylinder would behave like

$$q_1\gamma^{3/2}R^{1/2}\times(\text{a function of }\phi\text{, independent of }\gamma\text{ and }R).$$

A similar objection would apply if the inner cylinder were at rest. From here on it will therefore be assumed that neither cylinder is at rest.

It will also be supposed that the cylinders rotate in the same sense. With this provision, the speed of the unperturbed flow does not vanish at any point of the fluid and some simplification thereby results in the behaviour of the vorticity at large overall Reynolds numbers.

The general solution to the first order vorticity equation for which ζ_1 has the appropriate dependence of ϕ is given by (24) or, if $A = 0$, by (28). In both equations C, D are for the present to be regarded as undetermined constants. The behaviour of ζ_1 for large Reynolds numbers is thus governed (C, D apart) by the symptotic behaviour of $J_\mu(z)$ and $H^1_\mu(z)$, or, if $A = 0$, by the asymptotic behaviour of ρ^μ and $\rho^{-\mu}$.

When the cylinder speeds are such that $A \neq 0$ and $B \neq 0$, the behaviour of $J_\mu(z)$, $H^1_\mu(z)$ when both order and argument are large is required. With some minor modification due to substituting for the variables μ and z the more readily interpreted variables ρ, A, B and R, the asymptotic representations obtained by Debye may be used. An auxiliary variable σ is introduced which is related to the local speed

$$q_1 Q(\rho) = q_1\left(A\rho + \frac{B}{\rho}\right)$$

by

$$\sigma = \int^{\rho}\left(\frac{Q}{\rho}\right)^{1/2} d\rho\,. \tag{29}$$

The asymptotic expansions of $J_\mu(z)$, $H_\mu^1(z)$ for the values which z/μ assume in this problem may then be written (see Watson 1944, also Olver 1954)

$$\left.\begin{aligned} J_\mu(z) &\sim k_1\, e^{\sqrt{(Ri)}\sigma}\, P(R^{1/2}), \\ H_\mu^1(z) &\sim k_2\, e^{-\sqrt{(Ri)}\sigma} P(-R^{1/2}), \end{aligned}\right\} \tag{30}$$

where k_1, k_2 are constants independent of ρ, whose values need not be known here, and $P(R^{1/2})$ is an asymptotic series in $R^{1/2}$ whose coefficients are regular functions of ρ, A, B.

If the cylinder speeds are such that in the unperturbed motion $A = 0$ or $B = 0$, the behaviour of the vorticity component depends on the behaviour of $\rho^{\sqrt{(Ri+1)}}$, $\rho^{-\sqrt{(Ri+1)}}$ (see (28)) or of $J_1(\sqrt{(-Ri)}\rho)$, $H_1^1(\sqrt{(-Ri)}\rho)$. The asymptotic representations of these functions are analytically simpler than the asymptotic representations of $J_\mu(z)$, $H_\mu^1(z)$ when $A \neq 0$, $B \neq 0$ but are not otherwise importantly different. In fact, when $A = 0$ the asymptotic behaviour of $\rho^{\sqrt{(Ri+1)}}$, $\rho^{-\sqrt{(Ri+1)}}$ is correctly represented by the respective expressions for $J_\mu(z)$, $H_\mu^1(z)$ set out in (30) providing that $\log\rho$ is substituted for σ. Likewise, when $B = 0$, the asymptotic behaviour of $J_1(\sqrt{(-Ri)}\rho)$, $H_1^1(\sqrt{(-Ri)}\rho)$ is correctly represented by the above expressions for $J_\mu(z)$, $H_\mu^1(z)$ providing that ρ is substituted for σ. In either case, the function adopted for $\sigma(\rho)$ is related to the local speed of the unperturbed flow as in (29).

Thus, when the cylinders rotate in the same sense, the behaviour of the first order vorticity component at large Reynolds number may in every case be represented in the same asymptotic form : namely,

$$\zeta_1 \sim \mathscr{R}\{\kappa_1\, P(R^{1/2}) e^{\sqrt{(Ri)}\sigma + i\phi} + \kappa_2\, P(-R^{1/2}) e^{-\sqrt{(Ri)}\sigma + i\phi}\}, \tag{31}$$

where κ_1, κ_2 are constants independent of position, $P(R^{1/2})$ denotes an asymptotic series in $R^{1/2}$,and $\sigma(\rho)$ is related to the speed of the unperturbed motion as in (29).

It follows immediately from the way in which σ is related to the speed Q of the unperturbed motion, that $\sigma(\rho)$ is monotonic increasing; the complication that might have arisen if the speed of the unperturbed motion had vanished at some circle $\rho =$ constant between the cylinders having been avoided by supposing that both cylinders rotate in the same sense. The vorticity component ζ_1 thus resolves into two parts, both of which oscillate rapidly with changes in ρ, the amplitude of the first part decaying exponentially with increasing distance from the outer cylinder and the amplitude of the second part decaying exponentially with increasing distance from the inner cylinder. This 'exponential' property of the two parts that contribute to ζ_1 will be seen to play a significant role in determining the behaviour of the motion at large Reynolds number.

5. Behaviour of the first order stream function for large R

We now consider the behaviour for large R of the first order stream function ψ_1. It is to be anticipated that the behaviour of ψ_1 outside the boundary layers at each cylinder will differ from its behaviour inside these

layers. Accordingly the asymptotic behaviour of ψ_1 is derived firstly for points in the fluid whose distance from the nearest boundary is very much greater than $a/R^{-1/2}$ and secondly for points whose distance from the nearest boundary is $O(a/R^{-1/2})$.

It is convenient to resolve the solution for ψ_1 into the components

$$\psi_R = 2\mathscr{R}\{(CI_1 + DI_2)e^{i\phi}\},$$

$$\psi_I = A\rho^3 \cos\phi + 2\mathscr{R}\left\{\left(A_1\rho + \frac{B_1}{\rho}\right)e^{i\phi}\right\}.$$

The component ψ_R gives rise to the vorticity distribution ζ_1 and will be referred to as the strongly rotational component, whilst ψ_I makes no contribution to ζ_1 and will be referred to as the weakly rotational component. The strongly rotational component may further be divided into two parts, namely

$$\psi_{R1} = 2\mathscr{R}\{CI_1\, e^{i\phi}\}, \qquad \psi_{R2} = 2\mathscr{R}\{DI_2\, e^{i\phi}\},$$

each corresponding to one of the two parts of ζ_1 distinguished above.

For the case where the cylinder speeds are such that $A \neq 0$, the strongly rotational terms ψ_{R1}, ψ_{R2} give rise to the respective vorticity distributions $2\mathscr{R}\{CJ_\mu(z)e^{i\phi}\}$, $2\mathscr{R}\{DH^1_\mu(z)e^{i\phi}\}$. The functions I_1 and I_2 which define the way in which ψ_{R1}, ψ_{R2} vary with ρ, are expressed in terms of simple integrals of the corresponding functions $J_\mu(z)$, $H^1_\mu(z)$, and their asymptotic behaviour is therefore related to that of $J_\mu(z)$, $H^1_\mu(z)$. The detailed definitions of I_1, I_2 were given in (20). Whence also

$$\left.\begin{aligned} I_1' &= \int_1^\rho J_\mu(z)\, d\rho + \frac{1}{\rho^2}\int_1^\rho \rho^2 J_\mu(z)\, d\rho, \\ I_2' &= \int_\beta^\rho H^1_\mu(z)\, d\rho + \frac{1}{\rho^2}\int_\beta^\rho \rho^2 H^1_\mu(z)\, d\rho. \end{aligned}\right\} \tag{32}$$

On substituting for $J_\mu(z)$ and $H^1_\mu(z)$ the asymptotic expansions (30) we find that

$$\left.\begin{aligned} I_1 &\sim \kappa_1\left\{a_1 R^{-1/2} + \sum_{n=2}^{\infty} a_n R^{-n/2}\right\}e^{\sqrt{(Ri)}\sigma}, \\ I_2 &\sim \kappa_2\left\{b_1 R^{-1/2} + \sum_{n=2}^{\infty} b_n R^{-n/2}\right\}e^{-\sqrt{(Ri)}\sigma}, \\ I_1' &\sim \kappa_1\left\{\sqrt{(i)}\sigma' a_1 + \sum_{n=2}^{\infty} a_n^* R^{-n/2}\right\}e^{\sqrt{(Ri)}\sigma}, \\ I_2' &\sim \kappa_2\left\{-\sqrt{(i)}\sigma' b_1 + \sum_{n=2}^{\infty} b_n^* R^{-n/2}\right\}e^{-\sqrt{(Ri)}\sigma}, \end{aligned}\right\} \tag{33}$$

where the coefficients a_n, a_n^*, b_n, b_n^* are independent of R. As was to be expected, in view of the corresponding behaviour of $J_\mu(z)$, both I_1 and I_1' are smaller at any point ρ than at the outer cylinder ($\rho = \beta$) by a factor of $O(e^{-\sqrt{(\frac{1}{2}R)}d_2})$ where $d_2 = \sigma(\beta) - \sigma(\rho) > 0$. Similarly, I_2 and I_2' are both smaller at any point ρ than at the inner cylinder ($\rho = 1$) by a factor of $O(e^{-\sqrt{(\frac{1}{2}R)}d_1})$, where $d_1 = \sigma(\rho) - \sigma(1) > 0$. The asymptotic representations of ψ_{R1}, ψ_{R2} and also

ψ_I now follow from (33) and the expressions (21), (22) for the constants A, B, C, D. Hence,

$$\psi_{R1} \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\sqrt{\left(\frac{\beta q_1}{iq_2}\right)} + \sum_{n=1}^{\infty} \alpha_n R^{-n/2}\right] e^{-\sqrt{(Ri)}\{\sigma(\beta)-\sigma(\rho)\}+i\phi}\right\}, \tag{34 a}$$

$$\psi_{R2} \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[-\frac{\beta q_2}{q_1\sqrt{i}} + \sum_{n=1}^{\infty} \beta_n R^{-n/2}\right] e^{-\sqrt{(Ri)}\{\sigma(\rho)-\sigma(1)\}+i\phi}\right\}, \tag{34 b}$$

$$\psi_I \sim -\frac{A}{\rho}(\rho^2-1)(\beta^2-\rho^2)\cos\phi + \\ + \frac{2}{R^{1/2}} \mathscr{R}\left\{\left(\frac{\beta}{(\beta^2-1)\sqrt{i}}\left[\left\{\frac{\beta^2 q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\frac{1}{\rho} - \left\{\frac{q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\rho\right] + \right. \\ \left. + \sum_{n=2}^{\infty} (l_n \rho + m_n \rho^{-1}) R^{-n/2}\right) e^{i\phi}\right\}, \tag{34 c}$$

where α_n, β_n are independent of R, and l_n, m_n are independent of R and ρ. It is evident from (34), that the constants A, B, C, D are such that the strongly rotational components ψ_{R1}, ψ_{R2} and the weakly rotational component ψ_I are each $O(R^{-1/2})$ at the bounding cylinders. The fact that the components ψ_{R1}, ψ_{R2} are transcendentally small save in thin layers of thickness $O(aR^{-1/2})$ at the inner and outer cylinders follows automatically from the ' exponential ' behaviour of these components.

In the special case where $A = 0$ the analysis may be retraced with only small changes, caused by replacing $J_\mu(z)$, $H_\mu^1(z)$ by ρ^μ, $\rho^{-\mu}$; and, on putting $\log \rho$ for σ, the results (34) are obtained unimpaired. The asymptotic representations (43) therefore hold for all relevant values of A.

The asymptotic representation of the first order stream function ψ_1 appropriate to points whose distance from either boundary is much greater than $aR^{-1/2}$ is evidently that given by (34) for ψ_I. For points whose distance from either cylinder is $O(aR^{-1/2})$, the strongly rotational component is significant. Consider the flow near the inner cylinder. The asymptotic representation of ψ_1 for a point $\rho = 1+\xi R^{-1/2}$, where ξ is independent of Reynolds number, clearly includes ψ_I and ψ_{R2} but not ψ_{R1}. Whence, from (34 b) and (34 c), we find after some rearrangement that for fixed ξ,

$$\psi_1 \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\left(1-\frac{\beta q_2}{q_1}\right)\xi + \frac{\beta q_2}{q_1\sqrt{i}}(1-e^{-\sqrt{i}\xi})\right] e^{i\phi}\right\} + \\ + \sum_{n=2}^{\infty} \mathscr{R}\{P_n e^{i\phi}\} R^{-n/2} + \sum_{n=2}^{\infty} \mathscr{R}\{Q_n e^{i\phi-\sqrt{i}\xi}\} R^{-n/2}, \tag{35}$$

where P_n and Q_n denote polynomials in ξ of orders n and $n-1$ respectively. In the same way the asymptotic representation of ψ_1 for a point $\rho = \beta - \eta R^{-1/2}$, where η is independent of R, near the outer cylinder includes ψ_1 and ψ_{R1} but not ψ_{R2}. Thus, from (34 a) and (34 c) we find that for fixed η

$$\psi_1 \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\left(1-\frac{\beta q_2}{q_1}\right)\eta - \sqrt{\left(\frac{\beta q_1}{iq_2}\right)}(1-e^{-\sqrt{(iq_2/\beta q_1)}\eta})\right] e^{i\phi}\right\} + \\ + \sum_{n=2}^{\infty} \mathscr{R}\{S_n e^{i\phi}\} R^{-n/2} + \sum_{n=2}^{\infty} \mathscr{R}\{T_n e^{i\phi-\sqrt{(iq_2/\beta q_1)}\eta}\} R^{-n/2}, \tag{36}$$

where S_n and T_n denote polynomials in η (different from before) of orders n and $n-1$ respectively.

The main features of the motion described by these expansions are two-fold. The first is that the stream function ψ_1 may be expressed both outside and, with suitably rescaled coordinates, inside the boundary layers as an asymptotic series in $R^{1/2}$. The zero order stream function ψ_0 is independent of R, so that similar expansions hold for $\psi_0+\gamma\psi_1$. The second feature is that the vorticity ζ_1 associated with ψ_1 is zero, to the accuracy of the asymptotic representation for ψ_1, in the region outside the boundary layers. The vorticity ζ_0 is constant, so that the vorticity $\zeta_0+\gamma\zeta_1$ associated with $\psi_0+\gamma\psi_1$ is again, to the accuracy of the asymptotic representation, uniform throughout the region outside the boundary layers. The leading terms of the expansions (34 c), (35), (36) for ψ_1 are those that would be obtained by the use of the boundary layer approximation. The leading term of ψ_1, in particular, represents the first order perturbation to the inviscid motion (that is, the flow at infinite Reynolds number). As expected, the speed of the inviscid flow is non-zero at the bounding cylinders being of amount $2\beta\gamma q_2\cos\phi$ at the inner cylinder and $2\gamma q_1\cos\phi$ at the outer cylinder.

The behaviour of the stream function $\psi_0+\gamma\psi_1$ outside the boundary layers is typical of more general closed motions. It was to be expected that $\psi_0+\gamma\psi_1$ would be expressible as an asymptotic series of the form $\sum\psi^{(n)}R^{-n/2}$ where the coefficients $\psi^{(n)}$ are independent of R, and further (Batchelor 1956) that the inviscid motion would have uniform vorticity. It now appears that the vorticity associated with the higher order terms of this expansion is also uniform. With the general assumption that the stream function behave at large Reynolds number R like $\sum\psi^{(n)}R^{-n/2}$, where the coefficients $\psi^{(n)}$ are independent of R, this property is true of any two-dimensional motion whose streamlines are closed and lie in a domain free of shear layers. The proof of this result entails only a trivial extension by induction of the argument due to Batchelor, and will not be given here (though it may perhaps be remarked that in the iteration, the fact that the vorticity associated with $\psi^{(n)}$ is uniform follows from the requirement that the pressure distribution associated with $\psi^{(n+2)}$ should be single-valued).

In the present problem the vorticity $\zeta_0+\gamma\zeta_1$ comprises a uniform distribution $2A$ and a distribution which outside the boundary layers becomes transcendentally small (i.e. $o(R^{-n})$, for arbitrarily large n) when the Reynolds number is large. This latter distribution, it will be recalled, divides into a part which decays 'exponentially' rapidly with increasing distance from the inner boundary and a part which decays 'exponentially' rapidly with distance from the outer boundary. Further, each of these parts represents the vorticity distribution, apart only from a constant term and a term which is transcendentally small, in the appropriate boundary layer. By regarding the vorticity distribution as a uniform field with two 'exponentially decaying' fields superposed, we gain a picture of the distribution which comprehends at once both the behaviour of the vorticity in the boundary layers and the result that the vorticity is uniform, apart from a transcendentally small term, outside the boundary layers.

The concept of vorticity decaying so rapidly with increasing distance from the generating surfaces as to become transcendentally small outside a thin layer is familiar, and, in the boundary layer approximation, fundamental. The possible occurrence of a component in the vorticity distribution which does not vary rapidly with distance is less well recognised. The occurrence of such a component is, however, *a priori* to be expected. For any solution of the vorticity equation for which the diffusion term vanishes identically may supply a component which does not change rapidly when the Reynolds number is large. In the case of a two-dimensional motion with closed streamlines whose stream function can be represented as above in the form $\sum \psi^{(n)} R^{-n/2}$ it appears that the *only* possible slowly varying component is one for which the diffusion term vanishes. This is also true of certain axisymmetric motions with closed streamlines (Batchelor 1956).

Concerning the mode of representation of the stream function ψ_1 it is interesting to note that the asymptotic expansions (34 c), (35), (36) may be embraced in a single representation which holds uniformly throughout the fluid. Such a representation is achieved by writing for ψ_I its asymptotic series (34 c) and for ψ_{R1}, ψ_{R2} their 'boundary layer' representations and then combining all the terms. Thus, we put

$$\psi_1 = -\frac{A}{\rho}(\rho^2-1)(\beta^2-\rho^2)\cos\phi + \frac{2\beta}{R^{1/2}(\beta^2-1)} \times$$
$$\times \left[\left\{\frac{\beta^2 q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\frac{1}{\rho} - \left\{\frac{q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\rho\right]\cos(\phi - \tfrac{1}{4}\pi) +$$
$$+ \sum_{n=2}^{m} \mathscr{R}\{(l_n \rho + m_n \rho^{-1}) e^{i\phi}\} R^{-n/2} + \sum_{n=2}^{m} \mathscr{R}\{Q_n e^{i\phi - \sqrt{i}\xi}\} R^{-n/2} +$$
$$+ \sum_{n=2}^{m} \mathscr{R}\{T_n e^{i\phi - \sqrt{(iq_2/\beta q_1)}\eta}\} R^{-n/2} + o(R^{-m/2}). \quad (37)$$

In the boundary layers, the weakly rotational component may be re-expressed in re-scaled coordinates, and (37) then transforms into (35), (36). Outside the boundary layers, the strongly rotational component is transcendentally small, and (37) then reduces to (34 c). A similar representation holds for the stream function $\psi_0 + \gamma\psi_1$. Both representations correspond with the division of the vorticity field into a uniform component and two 'exponentially decaying' components.

6. The higher order approximations of boundary layer theory

In the following, an expansion for ψ_1 which holds uniformly throughout the flow region is calculated by the methods of boundary layer theory. For simplicity attention is confined to the case in which $A = 0$. The key to the procedure to be used is the knowledge that ψ_1 can be expressed as in (37). Lagerstrom & Cole have proposed that a stream function may generally be

expressed (flows with separating boundary layers excepted) as a sum of components of the form $\sum_{n=0}^{\infty} \mathscr{I}_n(z_1, z_2) R^{-n/2}$ and $\sum_{n=0}^{\infty} \mathscr{R}_n(z_1, z_2) R^{-n/2}$, where z_1, z_2 are coordinates such that z_2 vanishes on the solid boundaries, and R is a Reynolds number. The second component is further supposed to be transcendentally small in the region outside the boundary layers. Once it has been established, or can be regarded as established, that the stream function may be expressed in this form it is not difficult to devise a procedure for calculating the higher order corrections to the Prandtl approximation. The aim in what follows is to illustrate this procedure for a motion which is closed.

We begin, then, by assuming that for large R,

$$\psi_1 = \sum_{n=0}^{m} \mathscr{I}_n(\rho, \phi) R^{-n/2} + \sum_{n=1}^{m} \mathscr{R}_{n1}(\xi, \phi) R^{-n/2} + \\ + \sum_{n=1}^{m} \mathscr{R}_{n2}(\eta, \phi) R^{-n/2} + o(R^{-m/2}), \quad (38)$$

the terms $R_{n1}(\xi, \phi)$ being supposed transcendentally small for large ξ (i.e. $o(\xi^{-n})$, for any n) and the terms $R_{n2}(\eta, \phi)$ being supposed transcendentally small for large η. The leading summation represents the (truncated) asymptotic series for ψ_1 appropriate to a fixed point in the fluid.

As was remarked in the preceding section, the vorticity associated with the terms $\mathscr{I}_n$ must be uniform throughout the motion. Thus

$$\nabla^2 \mathscr{I}_n = \delta_n, \quad (39)$$

where the δ_n are constants independent of position. To obtain equations for $R_{n1}(\xi, \phi)$ we consider the boundary layer at the inner cylinder. The contribution of the terms $\mathscr{R}_{n2}$ to ψ_1 may here be neglected. On expressing the variables that occur in the equations of motion (13) in terms of ξ, ϕ and then expanding them in powers of $R^{-1/2}$, we get

$$\left.\begin{aligned} &\frac{\partial^4 \mathscr{R}_{11}}{\partial \xi^4} - \frac{1}{\alpha^2} \frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^2 \partial \phi} = 0, \\ &\frac{\partial^4 \mathscr{R}_{21}}{\partial \xi^4} - \frac{1}{\alpha^2} \frac{\partial^3 \mathscr{R}_{21}}{\partial \xi^2 \partial \phi} = -\frac{2}{\alpha} \frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^3} + \frac{1}{\alpha^3} \frac{\partial^2 \mathscr{R}_{11}}{\partial \xi \partial \phi} - \frac{2\xi}{\alpha^3} \frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^2 \partial \phi}, \\ &\quad \cdots \cdots \cdots \cdots \cdots \cdots \end{aligned}\right\} \quad (40)$$

where $\alpha = 1$. The equations for $\mathscr{R}_{n2}(\eta, \phi)$ may be obtained in a similar way and, in fact, the first two equations are correctly given by (40) if R_{11}, R_{21}, ξ are replaced by R_{12}, R_{22}, $-\eta$ respectively and α is put equal to β. It may be noted that these equations for R_{n1}, R_{n2} contain no terms from the weakly rotational component of ψ_1. This would not be generally true. In general, the weakly rotational component will contribute to the convection velocity in the vorticity equation (though, of course, it will not in general contribute to the vorticity derivatives).

The boundary conditions for the coefficients $\mathscr{I}_n$ of the weakly rotational component are

$$\left.\begin{aligned} &\mathscr{I}_0(1,\phi)=0, \quad \mathscr{I}_0(\beta,\phi)=\text{constant}, \\ &\mathscr{I}_n(1,\phi)+\mathscr{R}_{n1}(0,\phi)=0, \ \mathscr{I}_n(\beta,\phi)+\mathscr{R}_{n2}(0,\phi)=\text{constant} \ \ (n\geqslant 1), \end{aligned}\right\} \tag{41}$$

whilst the boundary conditions for the coefficients $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ are

$$\left.\begin{aligned} \frac{\partial \mathscr{R}_{n1}(0,\phi)}{\partial \xi}+\frac{\partial \mathscr{I}_{n-1}(1,\phi)}{\partial \rho} &= 2\cos\phi \quad (n=0), \\ &= 0 \quad (n\geqslant 1), \\ \frac{\partial \mathscr{R}_{n2}(0,\phi)}{\partial \eta}-\frac{\partial \mathscr{I}_{n-1}(\beta,\phi)}{\partial \rho} &= 2\cos\phi \quad (n=0), \\ &= 0 \quad (n\geqslant 1), \\ \lim_{\xi\to\infty}\mathscr{R}_{n1}(\xi,\phi)=\lim_{\xi\to\infty}\frac{\partial \mathscr{R}_{n1}(\xi,\phi)}{\partial \xi} &= 0, \\ \lim_{\eta\to\infty}\mathscr{R}_{n2}(\eta,\phi)=\lim_{\eta\to\infty}\frac{\partial \mathscr{R}_{n2}(\eta,\phi)}{\partial \eta} &= 0. \end{aligned}\right\} \tag{42}$$

In addition, each of the $\mathscr{I}_n$, $\mathscr{R}_{n1}$ and $\mathscr{R}_{n2}$ and the associated pressure distributions must be periodic functions of ϕ. The conditions on $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ at the outer edges of the boundary layers result from supposing that these terms are transcendentally small outside the boundary layers. The boundary conditions for the $\mathscr{I}_n$ derive from the requirement that the normal component of velocity at the boundary associated with these terms should annul the normal component of velocity at the boundary associated with the terms $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$. The boundary conditions for the $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ arise from adjusting the tangential component of velocity associated with the terms $\mathscr{I}_n$ to the tangential velocity of the boundary.

The iterative solution of these equations is complicated by the fact that the equations that govern $\mathscr{I}_n$ ((39) and (41)) are incomplete. That is, if $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ are supposed known, $\mathscr{I}_n$ is undetermined to the extent of two constants. One of these constants is δ_n and represents the (uniform) vorticity associated with $\mathscr{I}_n$. The other, Γ_n say, arises because the fluid encircles a closed boundary, and may be defined as the circulation round the inner cylinder of the motion associated with $\mathscr{I}_n$.

In the present problem it is *a priori* obvious from the symmetry of the configuration that δ_n and Γ_n vanish. (This may be seen by considering the motion in which the sign of γ is reversed.) The indeterminancy in $\mathscr{I}_n$ is thus resolved immediately. To resolve the indeterminancy without recourse to a symmetry argument, it is necessary to pass to the equations governing $\mathscr{R}_{n+1,1}$, $\mathscr{R}_{n+1,2}$. On solving for $\mathscr{R}_{n+1,1}$ a condition on the tangential velocity distribution $\partial \mathscr{I}_n(1,\phi)/\partial\rho$ emerges. Similarly, on solving for $\mathscr{R}_{n+1,2}$ a condition emerges on the tangential velocity $\partial \mathscr{I}_n(\beta,\phi)/\partial\rho$. From these two conditions, we may then deduce that $\delta_n=0$, $\Gamma_n=0$, and so on.

The equations for $\mathscr{I}_n, \mathscr{R}_{n1}, \mathscr{R}_{n2}$ may otherwise be solved without further difficulty. For the leading terms we get

$$\left.\begin{aligned}
&\mathscr{I}_0 = 0, \qquad \mathscr{I}_1 = \frac{2}{\beta^2-1}\left[\frac{2\beta^2}{\rho} - (\beta^2+1)\rho\right]\cos(\phi - \tfrac{1}{4}\pi),\\
&\mathscr{I}_2 = \frac{2}{(\beta^2-1)^2}\left[\frac{4\beta^2(\beta^2+1)}{\rho} + (\beta^4 - 10\beta^2 + 1)\rho\right]\sin\phi,\\
&\mathscr{R}_{11} = -2\mathscr{R}\left\{\frac{1}{\sqrt{i}}e^{-\sqrt{i}\xi + i\phi}\right\},\\
&\mathscr{R}_{21} = -2\mathscr{R}\left\{\left(\frac{\xi^2}{2} + \frac{2\xi}{\sqrt{i}} + \frac{5\beta^2-1}{(\beta^2-1)i}\right)e^{-\sqrt{i}\xi + i\phi}\right.,\\
&\mathscr{R}_{12} = 2\beta\mathscr{R}\left\{\frac{1}{\sqrt{i}}e^{-\sqrt{i}(\eta/\beta) + i\phi}\right\},\\
&\mathscr{R}_{22} = -2\mathscr{R}\left\{\left(\frac{\eta^2}{2\beta} + \frac{2\eta}{\sqrt{i}} + \frac{\beta(\beta^2-5)}{(\beta^2-1)i}\right)e^{-\sqrt{i}(\eta/\beta) + i\phi}\right\}.
\end{aligned}\right\} \tag{43}$$

when these terms are combined together as in (38) they yield an expression for ψ_1 which holds to $O(R^{-1})$ uniformly throughout the fluid.

7. The terms of higher order in γ

For the sake of completeness it is worth noting that the higher order terms of the perturbation of the motion may be solved in essentially the same way as the first order terms.

When the equations governing the higher order terms ζ_n, ψ_n of the expansions for the vorticity component ζ and stream function ψ are examined, it is seen that each varies with ϕ according to the respective forms

$$\zeta_n = \sum_{s=0}^{n} g_{n,\,n-2s}\, e^{i(n-2s)\phi}, \qquad \psi_n = \sum_{s=0}^{n} f_{n,\,n-2s}\, e^{i(n-2s)\phi}. \tag{44}$$

The equations governing $g_{n,s}$, $f_{n,s}$ are then

$$g''_{n,s} + \frac{1}{\rho}g'_{n,s} - \left(RAsi + \frac{RBsi + s^2}{\rho^2}\right)g_{n,s} = RG_{n,s}, \tag{45 a}$$

$$f''_{n,s} + \frac{1}{\rho}f_{n,s} - \frac{s^2}{\rho^2}f_{n,s} = -g_{n,s} + F_{n,s}, \tag{45 b}$$

where $G_{n,s}$, $F_{n,s}$ are defined by

$$\left.\begin{aligned}
&\sum_{s=0}^{n} G_{n,\,n-2s}(\rho)e^{i(n-2s)\phi} = \frac{1}{\rho}\sum_{s=1}^{n-1}\left(\frac{\partial\psi_r}{\partial\phi}\frac{\partial\zeta_{n-r}}{\partial\rho} - \frac{\partial\psi_r}{\partial\rho}\frac{\partial\zeta_{n-r}}{\partial\phi}\right),\\
&\sum_{s=0}^{n} F_{n,\,n-2s}(\rho)e^{i(n-2s)\phi} = 4\rho\cos\phi\nabla^2\psi_{n-1} + 2\rho^2(2+\cos 2\phi)\nabla^2\psi_{n-2} +\\
&\qquad\qquad + 4\rho^3\cos\phi\nabla^2\psi_{n-3} + \rho^4\nabla^2\psi_{n-4}.
\end{aligned}\right\} \tag{46}$$

Further the boundary conditions are

$$\left.\begin{aligned}
&f_{n,s}(1) = 0, \quad f_{n,0}(\beta) = \text{constant}, \quad f_{n,s}(\beta) = 0 \quad (s \neq 0),\\
&f'_{n,s}(1) = (-1)^{n-1}, \quad f'_{n,s}(\beta) = -(-\beta)^n\,(q_1/q_2), \quad (s = \pm n),\\
&f'_{n,s}(1) = 0, \quad f'_{n,s}(\beta) = -(-\beta)^{n-2}(\beta^2-1) \quad (|s| < n),
\end{aligned}\right\} \tag{47}$$

together with the condition that each term of the expression for the pressure is singled-valued.

These equations for the component functions $f_{n,s}(\rho)$ of the higher order perturbations ψ_n are similar to the equations for the component term $f(\rho)$ of the first order stream function ψ_1, and may be solved in a similar manner. The only features which call for special treatment are the interaction terms $F_{n,s}(\rho)$ and $G_{n,s}(\rho)$ and the components of the vorticity that do not vary with ϕ. The interaction terms in the equations for $f_{n,s}$ depend on the components $f_{r,s}$ for $r = 1, 2 \ldots n-1$, and are determined iteratively. To determine the components of the vorticity that do not vary with ϕ, it is necessary to use the pressure condition. It is found on examining the momentum equations that the necessary and sufficient condition for the pressure to be single-valued is that

$$g'_{n,0} = \frac{R}{\rho} \int_1^\rho \rho G_{n,0}\, d\rho,$$

whence

$$g_{n,0} = \text{constant} + R \int_1^\rho \frac{1}{\rho} \int_1^{\rho'} \rho' G_{n,0}(\rho')\, d\rho' d\rho. \tag{48}$$

The corresponding term of the stream function, namely $f_{n,0}$, may now be determined in the same way as the other terms $f_{n,s}$, $s \neq 0$.

It is not proposed to investigate these equations further. Enough has been set down to show that the solution may be completed in essentially the same way as for the first order terms. (The homogeneous form of (45 a) now has the general solution $2CJ_{\mu^*}(z^*) + 2DH^1_{\mu^*}(z^*)$, where $\mu^* = \sqrt{(-RAsi,)}$ $z^* = \sqrt{(RBsi+1)}$ and $A \neq 0$). Moreover the solution may be expected to behave at large Reynolds numbers in much the same way as did the first order terms. In particular it may be tentatively inferred that for large R, the vorticity may be expressed as the sum of a constant term $\sum_{n=0}^{\infty} g_{2n,0}\, \gamma^{2n}$, and an infinity of terms which vary exponentially rapidly with R. It may also be tentatively inferred that the stream functions ψ_n may at large Reynolds numbers be expressed as in (38). If this proves true, the expansions of ψ, ζ in powers of γ may be expected to converge uniformly with respect to R, when γ is sufficiently small.

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The supersonic motion of an aerofoil through a temperature front

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SUMMARY

A solution is given for the sound waves generated when an infinite wedge passes at supersonic velocity through the plane interface between different media. The results are then applied to the particular case when the two media consist of the same perfect gas at different temperatures. It is observed that the flow can be deduced for any symmetrical aerofoil of infinite span by superposition, and the case of a double wedge is considered in detail.

INTRODUCTION

When an aerofoil passes through the surface separating two media, as for example into a cloud, or through a temperature front, certain transient effects on the lift and drag are to be expected during the transition from one regime of steady flow to another. The transient effects will be examined by the method used in an earlier paper (Craggs 1956) on the reflection of sound waves.

The method used is first to examine the motion due to an infinite wedge moving with constant velocity. The fluid motion may then be expected to possess dynamic similarity, the velocity distribution being the same at all times after the wedge has broken the surface, except for a scale factor depending on the depth of penetration. The solution for the wedge can then be extended to any symmetric aerofoil by the principle of superposition, within the limitations imposed by the linearized theory. It is assumed throughout that the fluid motion is sufficiently small for linearized theory to be adequate, and that the motion is strictly two-dimensional.

1. THE EQUATION OF SOUND WAVES WITH DYNAMIC SIMILARITY

Use polar coordinates (r, θ) based on the point where the apex of the wedge cuts the surface of separation of the two media. Let t be the time measured from the instant at which the apex breaks the surface. Let c be the local velocity of sound and s the condensation, measured by the proportional increase in density due to fluid motion. Then the fluid velocity $\mathbf{q}$ is given by

$$\frac{\partial \mathbf{q}}{\partial t} = -c^2 \nabla s, \tag{1}$$

and the equation of motion is

$$c^2\nabla^2 s = \partial^2 s/\partial t^2. \tag{2}$$

Make the assumption of dynamic similarity and use λ, θ as independent variables, then (2) becomes

$$\lambda^2(1-\lambda^2/c^2)\partial^2 s/\partial\lambda^2 + \lambda(1-2\lambda^2/c^2)\partial s/\partial\lambda + \partial^2 s/\partial\theta^2 = 0. \tag{3}$$

Equation (3) is the same as the equation arising in the linearized theory of steady three-dimensional flow when the cone-field method is used (Ward 1955, p. 136) and the treatment here is similar.

For $\lambda > c$ equation (3) is hyperbolic and can be reduced to the canonical form

$$\frac{\partial^2 s}{\partial\mu^2} - \frac{\partial^2 s}{\partial\theta^2} = 0 \tag{4}$$

by the substitution

$$\sec\mu = \lambda/c. \tag{5}$$

The general solution is then of the form

$$s = f(\mu-\theta) + g(\mu+\theta) \tag{6}$$

where f and g are arbitrary functions, and particular cases can be solved by the method of characteristics.

Equation (3) is elliptic for $\lambda < c$ and is reduced by the substitution

$$\lambda/c = \operatorname{sech}(-\nu) \tag{7}$$

to the form

$$\frac{\partial^2 s}{\partial\nu^2} + \frac{\partial^2 s}{\partial\theta^2} = 0. \tag{8}$$

Write s as the real part of a complex function w, where

$$w = s + i\tau, \tag{9}$$

then from (8)

$$w = w(\nu + i\theta) \tag{10}$$

and

$$\frac{\partial s}{\partial\nu} = \frac{\partial\tau}{\partial\theta}, \qquad \frac{\partial s}{\partial\theta} = -\frac{\partial\tau}{\partial\nu}.$$

2. The solution in the regions where the equation is hyperbolic

For simplicity, assume that the wedge moves in the direction normal to the surface separating the two media. Let the motion be in the direction $\theta = \frac{1}{2}\pi$, and let ρ_1, ρ_2 be the densities in the regions $0 > \theta > -\pi$ and $0 < \theta < \pi$, respectively. Use the suffixes 1 and 2 consistently to refer to the two media. Consider a plane with polar coordinates λ, θ (figure 1). Then in the regions $0 < \theta < -\pi$, $\lambda > c_2$ and $0 > \theta > -\pi$, $\lambda > c_1$ the equations are of hyperbolic type, the characteristics being the tangents to the sonic circles, $\lambda = c_2$, $\lambda = c_1$, respectively.

Consider first the motion of the wedge with velocity $V > c_1$ before the apex reaches the point $r = 0$. Then the motion is the familiar steady

motion of a wedge with supersonic velocity and the solution is well known. There is a weak shock wave at the Mach angle $\phi_1 = \sin^{-1}(c_1/V)$ to the direction of motion, attached at the apex of the wedge. In front of the shock wave the velocity of the fluid is zero and the condensation is also zero. Behind the shock wave the condensation is

$$s_1 = \sigma_1 = M_1 \delta \sec \phi_1, \tag{11}$$

where M_1 is the Mach number, $M_1 = V/c_1$, and δ is the semi-angle of the wedge.

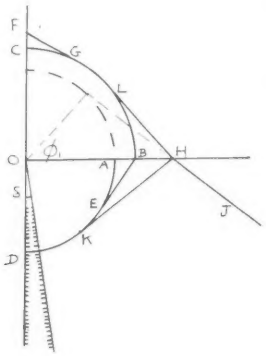

Figure 1.

To discuss the motion after the wedge has broken the surface, measure t from the instant at which the apex reaches the surface. Then the condition of steady motion gives the motion as $t \to 0+$, $\lambda \to \infty$. The boundary conditions on the equations as $\lambda \to \infty$ are therefore

$$\left.\begin{aligned} &s_1 = \sigma_1, && -\tfrac{1}{2}\pi < \theta < -\tfrac{1}{2}\pi + \phi_1, \\ &s_1 = 0, && -\tfrac{1}{2}\pi + \phi_1 < \theta < 0, \\ \text{and} \quad &s_2 = 0, && 0 < \theta < \tfrac{1}{2}\pi, \end{aligned}\right\} \tag{12}$$

with similar conditions for $\frac{1}{2}\pi < \theta < \frac{1}{2}(3\pi)$.

Use of characteristics theory shows that there is a weak shock wave, of trace

$$\lambda = c_1 \sec(\phi_1 - \theta) \tag{13}$$

and of strength σ_1. This is represented by the line HJ in figure 1.

It is convenient to assume that $c_1 \sec \phi_1 > c_2$, so that the point H, where the shock wave cuts $\theta = 0$ lies outside of the circle $\lambda = c_2$. (Only slight alteration of the argument is needed when this assumption is false, and for the application contemplated, in which the difference in properties between the two media is small, the assumption certainly holds.) Further simple deductions from the theory of characteristics are that there are weak shock waves HK, HL corresponding to the reflection and refraction

of the incident wave HJ at the interface, and a new shock wave FG attached at the apex of the wedge. The condensation behind FG is

$$s_2 = \sigma_2 = M_2 \delta \sec \phi_1, \tag{14}$$

where $\phi_2 = \sin^{-1}(c_2/V)$. The conditions behind HK, HL can be deduced from the shock wave equations and the continuity of pressure

$$\rho_1 c_1^2 s_1 = \rho_2 c_2^2 s_2, \tag{15}$$

and of normal velocity $\quad c_1^2 \partial s_1/\partial\theta = c_2^2 \partial s_2/\partial\theta, \tag{16}$

over the interface $\theta = 0$. The values are

$$s_1 = 2\sigma_1 \tan\phi_1 / \{\tan\phi_1 + k(m^2 \sec^2\phi_1 - 1)^{1/2}\} = \sigma_1' \tag{17}$$

and

$$\sigma_2' = m^2 k \sigma_1', \tag{18}$$

respectively, where $m = c_1/c_2$, $k = \rho_1/\rho_2$.

The condensations are now known throughout the regions $\lambda > c_1$, $\lambda > c_2$ except for the triangular region ABE, and in this region the theory of characteristics shows only that

$$s_1 = \sigma_1 + f_1(\theta - \mu), \tag{19}$$

the function g of equation (6) being here constant and equal to σ_1.

3. Regions where the equations are elliptic

To discuss the region $0 < \theta < \frac{1}{2}\pi$, $\lambda < c_2$ in which the equation for the condensation is elliptic, it is convenient to introduce a transformation

$$\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech} 2(\nu_2 + i\theta) \tag{20}$$

which maps the region on the upper half of the ζ plane. The closed boundary of the elliptic region maps into the axis $\eta_2 = 0$, and the points O, A, B, L, G, C of figure 1 correspond to the points

$$\zeta_2 = 0, m^2/(2-m^2), 1, m^2\sec^2\phi_1/(2-m^2\sec^2\phi), \sec^2\phi_2(2-\sec^2\phi_2), -1, \tag{21}$$

respectively.

Now consider the section AB of the interface. On the lower side of AB s_1 is governed by a hyperbolic equation, and from (19)

$$\frac{\partial s_1}{\partial\theta} = \frac{\partial s_1}{\partial\mu} = \lambda\left(\frac{\lambda^2}{c_1^2} - 1\right)^{1/2} \frac{\partial s_1}{\partial\lambda},$$

using (5). The continuity conditions (15) and (16) therefore give

$$\lambda(1-\lambda^2/c_1)^{1/2} \frac{\partial s_2}{\partial\lambda} = k\frac{\partial s_2}{\partial\theta} = -k\frac{\partial\tau_2}{\partial\nu} = -k\lambda\left(1-\frac{\lambda^2}{c^2}\right)^{1/2}\frac{\partial\tau_2}{\partial\lambda}, \tag{22}$$

using (7). This relation holds on the line AB, and on this line

$$\zeta_2 = \xi_2 = \lambda^2/(2c_2^2 - \lambda^2),$$

so

$$\frac{dw_2}{d\zeta_2} = \left\{1 - \frac{i}{m^2 k}\left(\frac{(2-m^2)\zeta_2 - 1}{1-\zeta_2}\right)^{1/2}\right\} F(\zeta_2) \tag{23}$$

on the line, where $F(\zeta_2)$ is real on the line, and by the principle of analytic continuation equation (23) holds throughout the region $\eta_2 > 0$.

The function $F(\zeta_2)$ is to be determined subject to the following conditions.

(i) Singularities of F are to be expected only at the points (21) and at at any zero of the function in brackets in (23).

(ii) For finite pressures, no singularity of $dw_2/d\zeta_2$ may be of higher order than a simple pole.

(iii) The point at infinity in the ζ_2 plane is an ordinary point, since it corresponds to an ordinary point, $\theta = \frac{1}{4}\pi$, $\lambda = c_2$ in the (λ, θ) plane.

(iv) From the conditions on the quarter circle BGC,

$$\frac{ds_2}{d\xi_2} = 0$$

for $\eta_2 = 0$, $-\infty < \xi_2 < -1$, $1 < \xi_2 < \infty$.

(v) The normal velocity on the surface of the wedge is constant, so $\partial\tau_2/\partial\theta = \partial s_2/\partial\lambda = 0$. On the corresponding section of the real axis $\eta_2 = 0$, $-1 < \xi_2 < 0$, the condition $\partial s_2/\partial\zeta_2 = 0$ holds.

(vi) A careful examination of the conditions in the immediate neighbourhood of the point A of figure 1, with use of the continuity conditions across the interface, shows that $F(\zeta_2)$ can contain only half-integer powers of the quantity $(2-m^2)\zeta_2 - m^2$.

A sufficiently general form for w_2 is then given by

$$\pi\frac{dw_2}{d\zeta_2} = \left[\frac{mk(1-\zeta_2)^{1/2} + i\{(2-m^2)\zeta_2 - m^2\}^{1/2}}{m^2k^2(1-\zeta_2) + (2-m^2)\zeta_2 - m^2}\right] \times \times \left[\frac{P_2 + Q_2\,\zeta_2^{-1/2}\{(2-m^2)\zeta_2 - m^2\}^{1/2}}{mk(1+\zeta_2)^{1/2}}\right], \quad (24)$$

with
$$P_2 = \frac{B}{(2-m^2\sec^2\phi_1)\zeta_2 - m^2\sec^2\phi_1} + \frac{C}{(2-\sec^2\phi_2)\zeta_2 - \sec^2\phi_2} \quad (25)$$

and
$$Q_2 = \frac{D}{(2-m^2\sec^2\phi_1)\zeta_2 - m^2\sec^2\phi_1} + \frac{E}{(2-\sec^2\phi_2)\zeta_2 - \sec^2\phi_2}. \quad (26)$$

For the solution in the region $\lambda < c_1$, $-\frac{1}{2}\pi < \theta < 0$, a similar transformation

$$\zeta_1 = \operatorname{sech} 2(\nu_1 + i\theta)$$

is used, and (24), together with the continuity conditions across the interface OA, leads to

$$m^2k\pi\frac{dw_1}{d\zeta_1} = \left[\frac{k\{1-(2m^2-1)\zeta_1\}^{1/2} - (1-\zeta_1)^{1/2}}{k^2\{1-(2m^2-1)\zeta_1\} - (1-\zeta_1)}\right] \times \times \left[\frac{P_1 + ikQ_1\,\zeta_1^{-1/2}\{1+(2m^2-1)\zeta_1\}^{1/2}}{m^2k(1+\zeta_1)^{1/2}}\right], \quad (27)$$

with
$$P_1 = \frac{B}{(2-\sec^2\phi_1)\zeta_1 - \sec^2\phi_1} + \frac{m^2C}{(2m^2-\sec^2\phi_2)\zeta_1 - \sec^2\phi_2} \quad (28)$$

and
$$Q_1 = \frac{D}{(2-\sec^2\phi_1)\zeta_1 - \sec^2\phi_1} + \frac{m^2E}{(2m^2-\sec^2\phi_2)\zeta_1 - \sec^2\phi_2}. \quad (29)$$

The constants B, C, D, E are easily determined by an examination of the singularities at the points corresponding to G, L, K of figure 1. At G, for example, where $\zeta_2 = \sec^2\phi_2/(2-\sec^2\phi_2)$, there is a discontinuity of magnitude σ_2 in s_2, whereas at the corresponding point in the ζ_1 plane, $\zeta_1 = \sec^2\phi_2/(2m^2-\sec^2\phi_2)$, there is no singularity. Applying the theory of residues in each plane, one obtains

$$2mk\sigma_2 = \frac{C+\sqrt{2}E(\sec^2\phi_2-m^2)^{1/2}\cos\phi_2}{mk\tan\phi_2-(\sec^2\phi_2-m^2)^{1/2}}$$

and
$$C-\sqrt{2}mkE\sin\phi_2 = 0,$$

whence
$$E = \sqrt{2}mk\sigma_2\sec\phi_2 \tag{30}$$

and
$$C = 2m^2k^2\sigma_2\tan\phi_2. \tag{31}$$

A similar argument based on the singularity in the ζ_2 plane at $\zeta_2 = m^2\sec^2\phi_1/(2-m^2\sec^2\phi_1)$, corresponding to H, where there is a discontinuity σ_2 in s_2, and on the singularity at the corresponding point K in the ζ_1 plane where s_1 changes by $(\sigma_1-\sigma_1')$, gives

$$B = -2m^4k^2\sigma_1\tan\phi_1 \tag{32}$$

and
$$D = -\sqrt{2}m^4k^2\sigma_1\sec\phi_1. \tag{33}$$

The functions $dw_2/d\zeta_2$, $dw_1/d\zeta_1$ are now determinate, and the solution of the problem is complete. The pressure at any point can be obtained by integration of (24) and (27) in terms of elliptic functions, but since in most cases only the pressure on the face of the wedge will be of interest, it is easier to evaluate the integrals numerically as required.

4. The special case of a temperature front

When the two media concerned are both composed of the same gas, but at different temperatures, some simplification is possible.

The gas laws then give $k = \rho_1/\rho_2 = T_2/T_1$, $c_1^2/c_2^2 = m^2 = T_1/T_2$, and $m^2k = 1$. It will generally be sufficient to use a first approximation, writing the absolute temperature ratio as $T_2/T_1 = 1+\epsilon$ and retaining only linear terms in ϵ. Then $k = 1+\epsilon$, $m^2 = 1-\epsilon$ and $\sec^2\phi_2 = \sec^2\phi_1(1+\epsilon\tan^2\phi_1)$. Equations (25), (26) and (30) to (33) then lead to

$$P_2 = \frac{4M_1\delta\epsilon\zeta_2\sin\phi_1}{(\zeta_2\cos 2\phi_1-1)^2} \tag{34}$$

and

$$Q_2 = \frac{\sqrt{2}M_1\delta\epsilon\{(1+\zeta_2)+\sin^2\phi_1(\zeta_2\cos 2\phi_1-1)\}}{\cos^2\phi_1(\zeta_2\cos 2\phi_1-1)^2}, \tag{35}$$

and equation (24) gives

$$\frac{\pi}{M\delta\epsilon}\frac{ds}{d\alpha} = \frac{2\alpha\sin\phi}{(\alpha\cos 2\phi+1)^2(1-\alpha)^{1/2}} - \frac{(1-\alpha)^{1/2}}{(2\alpha)^{1/2}\cos^2\phi(1+\alpha\cos 2\phi)^2} + \\ + \frac{\tan^2\phi}{(2\alpha)^{1/2}(1-\alpha)^{1/2}(1+\alpha\cos 2\phi)} \quad (\theta = \tfrac{1}{2}\pi), \tag{36}$$

where
$$\alpha = -\xi = \lambda^2/(2c^2-\lambda^2) \tag{37}$$

and the suffixes have been dropped, since the use of M_1 for M_2 introduces errors only of smaller order than the terms retained. The corresponding result in the other medium is, from (27),

$$\frac{\pi}{M\delta\epsilon}\frac{ds}{d\alpha} = \frac{2\alpha\sin\phi}{(\alpha\cos 2\phi+1)^2(1-\alpha)^{1/2}} + \frac{(1-\alpha)^{1/2}}{(2\alpha)^{1/2}\cos^2\phi(1+\alpha\cos 2\phi)^2} - \\ - \frac{\tan^2\phi}{(2\alpha)^{1/2}(1-\alpha)^{1/2}(1+\alpha\cos 2\phi)} \qquad (\theta = -\tfrac{1}{2}\pi). \quad (38)$$

Integration of (36) gives

$$\frac{s_2-\sigma_2}{M\delta\epsilon} = \frac{\sin\phi\cos 2\phi}{\sin^3 2\phi} - \frac{2\sin\phi(1-\alpha^2)^{1/2}}{\pi\sin^2 2\phi(1+\alpha\cos 2\phi)} - \frac{2\sin\phi\cos 2\phi}{\pi\sin^3 2\phi}\times \\ \times\sin^{-1}\left(\frac{\alpha+\cos 2\phi}{\alpha\cos 2\phi+1}\right) + \frac{1}{4\cos^3\phi} - \frac{\alpha^{1/2}(1-\alpha)^{1/2}}{2^{1/2}\pi\cos^2\phi(1+\alpha\cos 2\phi)} - \\ - \frac{1}{2\pi\cos^3\phi}\tan^{-1}\left\{\cos\phi\left(\frac{2\alpha}{1-\alpha}\right)^{1/2}\right\} - \frac{\sin^2\phi}{4\cos^3\phi} - \\ - \frac{\sin^2\phi}{2\pi\cos^3\phi}\sin^{-1}\left\{\frac{1-\alpha(1+2\cos^2\phi)}{1+\alpha\cos 2\phi}\right\}, \quad (39)$$

and a similar result follows from (37).

5. Application to general symmetric aerofoils

In the linearized theory, superposition of results is allowable, and the motion of a general symmetric aerofoil can be obtained by regarding it as being made up of a number of wedges.

Define a function $G(\lambda)$, $-\infty < \lambda < \infty$, as follows. For $0 < \lambda < \infty$, the value of $\delta G(\lambda)$ is equal to the condensation, s_2, in the above solution on the line $\theta = \frac{1}{2}\pi$ at the point λ. For $-\infty < \lambda < 0$, the value of $\delta G(\lambda)$ is the condensation at $\theta = -\frac{1}{2}\pi$, λ.

Now consider a symmetric aerofoil defined by

$$-X(y) < x < X(y), \qquad -a < y < 0,$$

where x, y are cartesian coordinates moving with the aerofoil and based on its leading edge, and write

$$\psi(y) = -dx/dy$$

for the slope of the surface, assumed everywhere small.

The condensation on the aerofoil before the leading edge breaks the surface of separation of the media is the same as in steady motion in medium 1. When the aerofoil is part way through the interface the condensation due to that part which has passed into the second medium is given by the foregoing theory, but the flow due to the part which has not yet left the medium 1 is still the steady flow appropriate to medium 1. Consider the conditions at a time t, less than a/V, after the leading edge breaks the surface, then the foregoing arguments give, on the aerofoil,

$$s = \psi_0 G\left(\frac{Vt+y}{ct}\right) - \int_{-Vt}^{0}\{\psi(0)-\psi(y^*)\}\times \\ \times G\left(\frac{V^2t+Vy-Vy^*}{Vct+cy^*}\right)dy^* \quad (0 > y > Vt), \quad (40)$$

and

$$s = \psi_0 G\left(\frac{Vt+y}{ct}\right) - \int_{-Vt}^{0} \{\psi(0)-\psi(y^*)\} G\left(\frac{V^2t+Vy-Vy^*}{Vct+cy^*}\right) dy^* -$$

$$- M \sec\phi_1 \int_{y}^{-Vt} \{\psi(0)-\psi(y^*)\}\, dy^* \quad (-Vt > y > -a), \quad (41)$$

where V is, as before, the (supersonic) velocity of the aerofoil. When the aerofoil has completely passed the interface only the expression (40) is required.

6. Numerical results for a double wedge passing through a temperature front

As an example, consider the motion through a temperature front of a double wedge, of angle δ and length $2b$.

λ	H	$\int^{1\cdot0} H\, d\lambda$	λ	H	$\int^{1\cdot0} H\, d\lambda$
1·0 to 1·5	−0·1341	0·00000	−0·05	−0·3978	−0·58093
0·95	−0·4896	−0·01559	−0·10	−0·3775	−0·60031
0·90	−0·5867	−0·04250	−0·15	−0·3567	−0·61866
0·85	−0·6328	−0·07299	−0·20	−0·3359	−0·63598
0·80	−0·6557	−0·10520	−0·25	−0·3153	−0·65226
0·75	−0·6654	−0·13823	−0·30	−0·2948	−0·66751
0·70	−0·6636	−0·17145	−0·35	−0·2744	−0·68174
0·65	−0·6570	−0·20447	−0·40	−0·2532	−0·69493
0·60	−0·6461	−0·22704	−0·45	−0·2415	−0·70730
0·55	−0·6322	−0·26900	−0·50	−0·2139	−0·71868
0·50	−0·6167	−0·30022	−0·55	−0·1939	−0·72888
0·45	−0·5960	−0·33054	−0·60	−0·1779	−0·73818
0·40	−0·5822	−0·36000	−0·65	−0·1535	−0·74646
0·35	−0·5623	−0·38861	−0·70	−0·1335	−0·75363
0·30	−0·5424	−0·41623	−0·75	−0·1124	−0·75978
0·25	−0·5223	−0·44284	−0·80	−0·0935	−0·76493
0·20	−0·5017	−0·46844	−0·85	−0·0738	−0·76911
0·15	−0·4810	−0·49301	−0·90	−0·0539	−0·77230
0·10	−0·4602	−0·51654	−0·95	−0·0334	−0·77449
0·05	−0·4399	−0·53904	−1·00 to $-\infty$	−0·0000	−0·77532
0·00	−0·4188	−0·56051			

Table 1.

It is convenient to tabulate a function $H(\lambda)$ defined by

$$H(\lambda) = \frac{G(\lambda)-\sigma_1}{\epsilon\sigma_1}.$$

Then (39) leads to the results in table 1.

The drag on the aerofoil then follows from (40) and (41). In table 2 a correction factor e is given such that the total drag is

$$\rho c_1^2 \sigma_1 b\delta\{2-e\epsilon\}. \quad (42)$$

The correction factor is tabulated against Vt/b, that is the number of half lengths by which the leading edge has passed the interface.

Vt/b	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5	6
e	0·2808	0·2752	0·3103	0·5476	0·0817	−0·0209	0·1712	0·2682

Table 2.

When Vt/b exceeds 6, the transient effects no longer influence the drag and the drag is constant and equal to that for steady motion in the new medium.

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On the generation of surface waves by shear flows

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SUMMARY

A mechanism for the generation of surface waves by a parallel shear flow $U(y)$ is developed on the basis of the inviscid Orr–Sommerfeld equation. It is found that the rate at which energy is transferred to a wave of speed c is proportional to the profile curvature $-U''(y)$ at that elevation where $U = c$. The result is applied to the generation of deep-water gravity waves by wind. An approximate solution to the boundary value problem is developed for a logarithmic profile and the corresponding spectral distribution of the energy transfer coefficient calculated as a function of wave speed. The minimum wind speed for the initiation of gravity waves against *laminar* dissipation in water having negligible mean motion is found to be roughly 100 cm/sec. A spectral mean value of the sheltering coefficient, as defined by Munk, is found to be in order-of-magnitude agreement with total wave drag measurements of Van Dorn. It is concluded that the model yields results in qualitative agreement with observation, but truly quantitative comparisons would require a more accurate solution of the boundary value problem and more precise data on wind profiles than are presently available. The results also may have application to the flutter of membranes and panels.

1. INTRODUCTION

The primary aim of the following analysis is the prediction of the energy transfer from a prescribed two-dimensional parallel shear flow in an inviscid incompressible fluid to a surface wave of prescribed wavelength and wave speed. The principal result to be obtained is that this transfer is proportional to the curvature of the velocity profile at that point in the profile where the mean air speed is equal to the wave speed. The formulation, as developed in § 3–5, is valid for surface waves of a rather general type and may be of some importance in the flutter of thin panels or membranes, but we shall focus our attention primarily on two-dimensional deep-water gravity waves.

We shall not attempt to develop the mechanism by which the surface wave energy is dissipated except in the initial formation of gravity waves on a previously smooth surface, and even in this case it will be necessary to introduce assumptions which would appear more questionable than those implied in the original model for energy transfer from the air to the surface wave.

Previous models of wave formation have been critically reviewed by Ursell (1956). The mathematical (as opposed to empirical) models may be briefly characterized as follows.

(1) *The Kelvin–Helmholtz model* (Lamb 1945, § 268) assumes a tangential discontinuity between uniform flows in air and water and predicts a critical wind speed of 650 cm/sec for the initiation of surface waves.

(2) *Jeffreys'* (1925) *sheltering model* (Lamb 1945, § 348) assumes a periodic component of wind stress in phase with the wave slope, which is supposed ot originate in the separation of the air flow over the wave crests and is described by a ' sheltering coefficient '. No method of calculating this coefficient is given, however, and the form of the results would be the same for other mechanisms.

(3) *The laminar flow models of Wuest* (1949) *and Lock* (1954) assume laminar flow in both air and water and yield the critical wind speed as in the corresponding analyses of stability for channel and boundary-layer flows.

(4) *Eckart's* (1953 b) *stochastic model* assumes an ensemble of gusts having prescribed durations in time and space. The details do not permit brief summary, but it should be noted that the assumed wind structure differs widely from that considered here. It may approximate a storm more closely, but it would be far more difficult to reproduce in a controlled experiment.

The model to be developed here improves on the Kelvin–Helmholtz model by allowing for distributed, rather than concentrated vorticity*, resembles Jeffreys' model in predicting an equivalent value for his sheltering coefficient, and is similar to the investigations of Wuest and Lock inasmuch as it leads to a boundary-layer stability problem. We define this model by the following assumptions.

(*a*) *The air.* We assume the air to be inviscid and incompressible and, in the absence of the wave motion, to have the prescribed mean shear flow $U(y)$. The subsequent disturbances in velocity and pressure associated with the wave motion will be assumed two-dimensional and sufficiently small to justify the linearization of the equations of motion. Turbulent fluctuations, albeit decisive in maintaining the mean shear flow, are neglected in the perturbation equations. The end result of these assumptions is the inviscid Orr–Sommerfeld equation (3.4).

(*b*) *The water.* We assume the motion of the water to be inviscid, incompressible, and irrotational, and the slope of the displaced surface to be small. Following Jeffreys (1925; Lamb 1945, § 348), the dissipation in the water will be calculated in the second approximation as a perturbation on the inviscid flow, assuming laminar

* The assumption of a concentrated vortex sheet in the Kelvin–Helmholtz model was criticized originally by Rayleigh (1880; also 1945, § 366).

motion and a free surface, but this result will appear only in the calculation of minimum wind speed (§ 6)*. We shall neglect any mean motion of the water that may be induced by the mean air flow. This neglect could be justified if either the actual mean velocity were small compared with the wave speed (so that it could only produce negligible inertial forces) or were laminar and confined to a layer thin compared with the wavelength.

(*c*) *Wave speed.* We assume that the access of inertia associated with the disturbance in the air has a negligible effect on the magnitude of the surface wave speed, only that component of the aerodynamic force that is in phase with the wave slope being regarded as important. This assumption, which is introduced only to simplify the algebraic operations by virtue of an expansion in the specific gravity of the air (and which is identical with that made in Jeffreys' sheltering theory), rules out the Kelvin–Helmholtz mechanism of wave formation *a priori*.

Referring to the assumption of an inviscid fluid, an appropriate Reynolds number for travelling waves in a slightly viscous fluid is

$$R = |U-c|/k\nu = |U-c|\lambda/2\pi\nu, \tag{1.1}$$

where U, c, λ, ν, and k denote the speed of the undisturbed flow, wave speed (phase velocity), wavelength, kinematic viscosity, and wave number. R may be evaluated for either air or water, but it will attain its smallest values in the air, both because of the higher kinematic viscosity and because $U-c$ may tend to zero. In the examples treated subsequently (§ 6, § 7) R will have values of the order of 10^3 at the outer edge of the boundary layer, though it tends to zero at the critical point where $U = c$. This implies that viscous forces in the air are likely to be important only in the neighbourhood of $U = c$, where their omission leads to a singularity in the equations of motion (§ 3). Previous studies of channel and boundary-layer flows (Lin 1955, ch. 5) suggest that this singularity provides an adequate representation of viscous effects (on *growing* waves) for sufficiently large average values of R. We specifically remark, on the other hand, that (in contrast to the Poiseuille or boundary-layer stability problems) the perturbation inertia forces for sufficiently large values of ck/ν may be expected to dominate the perturbation friction forces in the immediate neighbourhood of the boundary by virtue of the surface wave there. This implies, in particular, that the viscous drag forces of the air are negligible compared with the normal pressure in their effect on the surface wave (cf. preceding footnote).

* It might be thought that the air, behaving as in the boundary layer over an oscillating plane (cf. Lamb 1945, § 351) and having the higher kinematic viscosity, could contribute more damping than the water acting as a free surface (Lamb 1945, § 348). The ratio of the damping coefficients for small, laminar motions is found to be $(\mu_a/4\mu_w)[c^4/2g\nu_a(\overline{c-U})]^{\frac{1}{2}}$, where c denotes wave speed, ν kinematic viscosity, μ true viscosity, and $\overline{c-U}$ an appropriate average through the region of unsteady viscous influence. This ratio is about 0·1 in the calculation of the minimum wind speed (§ 6).

The foregoing arguments appear sufficient to justify an inviscid model for the determination of a first approximation to the disturbed motion of the air and the consequent energy transfer to the wave, but they throw no light on the perturbation Reynolds stresses associated with the interaction between turbulent fluctuations in the original and perturbed flows (see Appendix). It is implicit in our model that these also are negligible compared with the perturbation inertia forces except in the neighbourhood of $U = c$. It would be difficult to provide an adequate *a priori* justification of this hypothesis, but it appears reasonable for gravity waves in consequence of their relatively long wavelength.

It perhaps may be regarded as obvious that, on the one hand, models such as we consider here represent a gross simplification of the real problem of generation of surface waves by wind and cannot be expected to yield quantitative results except under very special circumstances ; but that, on the other hand, our present very limited knowledge of turbulent flows makes substantial theoretical progress unlikely without appeal to simplified models. The ranges of validity of such models must be established primarily by comparison with experimental data; this is scarcely available in sufficiently detailed and reliable form to warrant firm conclusions at the present time, but it seems unlikely that any single model (for which mathematical analysis is feasible) will prove adequate for all circumstances*.

2. Equation of motion for surface waves

We consider first the equation governing the propagation of a small, but otherwise rather general type of surface wave that gives rise to an aerodynamic pressure p_a acting on the surface. Let m be the effective mass per unit area and L a linear operator such that $L\eta$ gives the stress resisting a deformation η of the surface; then the equation of motion reads

$$L\eta + m\eta_{tt} = -p_a. \tag{2.1}$$

We assume the surface wave to have the form

$$\eta(x, t) = ae^{ik(x-ct)}, \tag{2.2}$$

where a denotes the amplitude, k the wave number, c the phase velocity, x a coordinate measured in the direction of propagation, and t the time; here and subsequently, following the usual convention, the real parts of complex quantities are implied in the final interpretation. We also assume the aerodynamic pressure to be represented by

$$p_a = (\alpha + i\beta)\rho_a U_1^2 k\eta, \tag{2.3}$$

* " How does wind acting on water give rise to waves ? This is a question which has never been satisfactorily answered in detail. This is partly because we do not know the exact constitution of an ocean wave, nor of the wind, nor the exact way in which the one acts upon the other. Notwithstanding our ignorance of the details, the general nature of the operation has been fairly well made out." (Durand 1896). Ursell, in his (1956) survey of the problem, opens with the statement that " wind blowing over a water surface generates waves in the water by a physical process which cannot be regarded as known " and concludes that " the present state of our knowledge is profoundly unsatisfactory ".

where ρ_a denotes the density of the air, U_1 an as yet arbitrary reference speed for the air, $k\eta$ (in magnitude) the local slope of the wave, and $\alpha+i\beta$ a dimensionless pressure coefficient; in general, α and β are functions of both c and k which depend on the solution to the aerodynamic boundary value problem (§ 3). Substitution of (2.2) and (2.3) in (2.1) yields

$$L\eta - mk^2c^2\eta = -(\alpha+i\beta)\rho_a U_1^2 k\eta, \tag{2.4}$$

which is an eigenvalue equation relating c and k for the assumed wave motion.

The operator L may be eliminated from (2.4) by referring it to the *free surface wave speed* c_w in the absence of the aerodynamic pressure. In the latter event (2.4) reduces to

$$L\eta = mk^2c_w^2\eta, \tag{2.5}$$

which implies that $mk^2c_w^2$ is the eigenvalue of the operator L. Substituting (2.5) in (2.4), we may place the result in the form

$$c^2 = c_w^2 + s(\alpha+i\beta)U_1^2, \tag{2.6}$$

where s denotes the relative mass parameter

$$s = \rho_a/mk. \tag{2.7}$$

We emphasize that (2.6) does not represent an explicit solution for c inasmuch as α and β exhibit an implicit dependence thereon.

We turn now to the special case of gravity waves on deep water of density ρ_w. The effective mass then is ρ_w/k, while the operator L is simply $\rho_w g$, and we have (cf. Lamb 1945, § 228, § 229)

$$c_w^2 = g/k, \qquad s = \rho_a/\rho_w. \tag{2.8 a, b}$$

We remark that in the Kelvin–Helmholtz model (Lamb 1945, § 268) $\alpha+i\beta = -1$ if U_1 is chosen as $U-c$, where U is the (constant) air speed; in this approximation the aerodynamic pressure is directly proportional to, but 90° out of phase with, the wave slope and represents an access of inertia. In Jeffreys' model (Lamb 1945, § 348), on the other hand, only the component of aerodynamic pressure in phase with the wave slope is considered, and β corresponds to his sheltering coefficient if $U_1 = U-c$.

We shall attempt a solution of the eigenvalue equation (2.6) only for those values of c and k that render the magnitude of the third term small compared with c_w^2, on which hypothesis we may take $c = c_w$ as a zeroth approximation, evaluate $\alpha+i\beta$ with $c = c_w$, and take

$$c = c_w[1+\tfrac{1}{2}s(\alpha+i\beta)(U_1/c_w)^2] \tag{2.9}$$

as a first approximation. If we define the *negative damping ratio* ζ as the energy rate of growth *per radian**, the first approximation to ζ_a (a denoting air) is given by (now dropping the subscript w from c, as is permissible in this approximation)

$$\zeta_a = 2\mathscr{I}\{c\}/\mathscr{R}\{c\} = s\beta(U_1/c)^2. \tag{2.10}$$

Our primary problem, then, is the determination of β.

* The corresponding logarithmic decrement is $-\pi\zeta$, while the rate of growth of η *per unit time* is $\frac{1}{2}\zeta kc$.

The approximations (2.9) and (2.10) correspond to what we may designate as *weak coupling* of the two fluids. In the Kelvin–Helmholtz model, on the other hand, we have *strong coupling*, and at the transition from stability to instability [$U = U_m$, $s \ll 1$, see Lamb 1945, § 268 (4)]

$$s(U_m/c_m)^2 = 1, \tag{2.11}$$

where c_m (which depends on surface tension, as well as gravity) is the minimum value of c_w. We infer from this that the approximation (2.9) may be regarded as valid only if

$$|\alpha + i\beta|(U_1/U_m)^2 \ll (c/c_m)^2. \tag{2.12}$$

The fact that waves do begin to form at wind speeds very much less than the Kelvin–Helmholtz value U_m implies that the inequality (2.12) may be assumed in describing incipient wave formation. Of course, (2.6) could be solved without further approximation, but the boundary value problem to be solved (§ 3) would be far more difficult in consequence of the fact that the dependence of c on k therein could not be specified *a priori*.

3. Equations of motion for the air

The equations of motion governing a small perturbation of a two-dimensional shear flow $U(y)$ in an inviscid incompressible fluid of density ρ_a are (see Appendix)

$$\rho_a(u_t + Uu_x + vU_y) = -p_x, \tag{3.1 a}$$

$$\rho_a(v_t + Uv_x) = -p_y, \tag{3.1 b}$$

$$u_x + v_y = 0, \tag{3.1 c}$$

where u and v denote the x- and y-components of the perturbation velocity, p the perturbation pressure, and subscripts partial differentiation. Introducing a stream function according to

$$u = -\psi_x, \qquad v = \psi_y, \tag{3.2 a, b}$$

and assuming (by virtue of linearity) ψ and p to exhibit the same dependence on x and t as η in (2.2), we obtain

$$\rho_a[(U-c)\psi_y - U_y\psi] = p, \tag{3.3 a}$$

$$\rho_a k^2(U-c)\psi = p_y. \tag{3.3 b}$$

Elimination of p then yields

$$(U-c)\psi_{yy} - [k^2(U-c) + U_{yy}]\psi = 0. \tag{3.4}$$

Equation (3.4), which was studied originally by Rayleigh (1880; also 1945, § 366 (7)), is the inviscid form of the well known Orr–Sommerfeld equation (Lin 1955, (1.3.15)). We observe that the omission of the viscous terms leads to a (regular) singularity at $U = c$.

The subsequent analysis will be carried out in terms of the dimensionless variables ξ, w, and ϕ, defined by

$$\xi = ky, \qquad U - c = U_1 w(\xi), \qquad \psi = U_1 \phi(\xi)\eta(x,t), \tag{3.5 a, b, c}$$

where U_1 is the arbitrary reference velocity introduced in (2.3). Introducing these in (3.4), we obtain

$$\phi'' - [1 + (w''/w)]\phi = 0. \tag{3.6}$$

The boundary conditions to be imposed on ϕ are dictated by the requirements that the interface (originally $y = y_0$) shall remain a streamline and that the disturbance shall die out at infinity. Noting that the horizontal velocity relative to the wave is approximately $U - c$, we obtain for the first condition

$$\psi_x/(U-c) = ik\eta \quad \text{at } y = y_0 + \eta \doteqdot y_0. \tag{3.7}$$

We remark that the approximation $u - c = U - c$ assumes only that the slope of the wave is small (amplitude small compared with wavelength, that is, $ka \ll 1$). If y were interpreted as the distance above a fixed plane it also would be necessary (in order to linearize the boundary condition) to assume the amplitude to be small compared with the length of any region in the neighbouring flow over which local variations must be considered. This would be a rather severe restriction in consequence of the large velocity gradients near the boundary, but it may be avoided by assuming y to specify the streamline that would have been a distance y above the undisturbed interface and applying the boundary condition at $y = y_0$. Perhaps the most satisfactory proof of this statement is to replace the independent variables x and y in (3.1 a, b, c) by x and Ψ, where Ψ is the total (as opposed to perturbation) stream function (von Mises transformation), write $U = U(\Psi)$ rather than $U(y)$, linearize with respect to the velocity perturbations about $U = U(\Psi)$, and apply the boundary condition (3.7) at $\Psi = \Psi_0$, where $U_0 = U(\Psi_0)$. The form of the equations is simpler in terms of y, however, and we shall retain it with the implicit understanding that it specifies a given streamline, rather than distance from a fixed plane. We also note that the boundary condition may be applied at any streamline, $y = y_0$, for which $ky_0 \ll 1$; thus, U_0 need not vanish, and it is not necessary to prescribe the shear profile all the way to the surface. In effect, we assume that a small transition layer, within which $0 \leqslant U \leqslant U_0$, moves with the surface wave*.

Introducing the dimensionless variables of (3.5), the condition (3.7) becomes

$$\phi_0 = w_0, \tag{3.8 a}$$

where the zero subscripts imply $\xi = \xi_0$. The requirement that the disturbance die out at infinity yields simply

$$\phi \to 0 \quad \text{as } \xi \to \infty. \tag{3.8 b}$$

The perturbation pressure is given by (3.3 a) or, in terms of the dimensionless variables, by

$$p = \rho_a U_1^2 k(w\phi' - w'\phi)\eta. \tag{3.9}$$

* The assumptions $ky_0 \ll 1$ and $|U-c| \gg v$ alone, without further linearization or the neglect of the dominant viscous forces, imply that p and v may be assumed constant across the transition layer within the approximations of Prandtl's boundary-layer theory, as applied to the *disturbed* flow.

Comparing (3.9) with (2.3) at $\xi = \xi_0$ and invoking (3.8 a) we obtain

$$\alpha + i\beta = w_0(\phi_0' - w_0'). \tag{3.10}$$

This completes the formulation of the aerodynamic boundary value problem, which now is stated by (3.6), (3.8 a, b), and (3.10).

4. Implicit solution

We next obtain an implicit integral expression for $\alpha + i\beta$, both in order to illuminate the decisive role of the singularity at $U = c$ ($w = 0$) and as the first step in an approximate determination of β. Multiplying both sides of (3.6) by ϕ^*, the complex conjugate of ϕ, integrating from $\xi = \xi_0$ to $\xi = \infty$, integrating $\phi''\phi^*$ by parts, and imposing the boundary conditions (3.8 a, b), we obtain

$$\int_{\xi_0}^{\infty} \{|\phi'|^2 + [1 + (w''/w)]\,|\phi|^2\}\, d\xi = [\phi^*\phi']_0^{\infty} = -w_0\,\phi_0'. \tag{4.1}$$

Now, from (3.10), since w is approximately real, β is given by the imaginary part of $w_0\,\phi_0'$, and the only contribution of the integral in (4.1) to this imaginary part must come from the singularity at $w = 0$. It follows that

$$\beta = -\mathscr{I}\left\{ \int_{\xi_0}^{\infty} |\phi|^2 (w''/w)\, d\xi \right\}. \tag{4.2}$$

The path of integration in (4.2) must be indented either over or under the singularity at $\xi = \xi_c$, where $w(\xi_c) = 0$, and on this choice depends the sign of β. This rather delicate and evidently decisive question has been discussed in some detail by Lin (1955, § 4.3 and ch. 8), and we merely state the conclusion that the path must be indented under the singularity*. Applying the calculus of residues then yields

$$\beta = -\pi |\phi_c|^2 (w_c''/w_c'), \quad w_c = 0, \tag{4.3}$$

where the subscript c implies evaluation at $\xi = \xi_c$.

The result (4.3) implies that, *in the absence of dissipative forces*, a motion of the type (2.2) will be stable or unstable according as the curvature of the wind profile (U'') at that elevation where the wind speed is equal to the wave speed is positive or negative, respectively. We infer from this result that:

(1) only those waves having speeds in that range of the wind profile for which $-U''$ is large may be expected to grow; the lower limit for c may be imposed by the existence of a sub-layer of linear profile or by the interaction of the waves with the wind profile, while the upper limit will be rather less than the wind speed outside of the boundary layer;

(2) in the *initial* phases of wave formation, those waves having speeds well down in the profile (large $-U''$) may be expected to predominate;

* This assumes $c = c_w$ is real. In the next approximation $\mathscr{I}\{c\} > 0$, so that the singularity lies slightly above the real axis (assuming $w_c''/w_c' < 0$), and the path of integration in (4.2) passes under the singularity without the necessity of indentation.

(3) experimental measurements of aerodynamic forces on stationary wave models (Stanton *et alia* 1932; Motzfield 1937; Thijsse 1951) may not yield significant values of such parameters as Jeffreys' sheltering coefficient, since the point at which $U = c$ then occurs right at the boundary.

We remark that the result (4.3) is directly related to Taylor's (1915) theorem that the momentum transfer from mean to disturbed flow in a shear profile is proportional to $-U''$; alternatively, the result may be interpreted in terms of the vortex theory of instability developed by Lin (1955, § 4.4), following earlier ideas of von Kármán (see also Rayleigh 1945, vol. 2, p. 391).

We now proceed to develop an integral expression for ϕ_c which may be used for an approximate determination of β in conjunction with (4.3). We start from (3.6) in the modified form

$$(w\phi' - w'\phi)' = w\phi. \tag{4.4}$$

Integrating both sides of this equation between $\xi = \xi_c$ and $\xi = \infty$, imposing the boundary condition (3.8 b) at $\xi = \infty$, and noting that

$$\lim_{\xi \to \xi_c} w\phi' = 0 \tag{4.5}$$

by virtue of the fact that ϕ' can have only a logarithmic singularity at the singular point of the differential equation*, we obtain

$$\phi_c = \frac{1}{w_c'} \int_{\xi_c}^{\infty} w\phi \, d\xi. \tag{4.6}$$

The principal attribute of (4.6), aside from the smoothing effect intrinsically associated with integration, is that it is locally insensitive to errors in (the approximation to) ϕ in the neighbourhood of the singular point $w = 0$.

We conclude this section by remarking that the approximations introduced up to this point may be regarded as inherent in our model. Additional approximations will be made in the following sections in order to simplify the mathematical analysis, but these are, at least in principle, subject to improvement without modification of the model.

5. Approximate solution for β

The exact solution of the differential equation (3.6) in terms of known functions for typical velocity profiles does not appear to be possible. The usual (Heisenberg) expansion in the parameter k^2 for the problem of thin (in wavelengths) boundary layers (Lin 1955, § 5.5) is not well suited to the relatively thick turbulent boundary-layer profiles that are of interest in the present problem.

* It may be proved that

$$\phi = \phi_c[1 + (w_c'' w_c')(\xi - \xi_c)\log(\xi - \xi_c) + O(\xi - \xi_c)]$$

in the neighbourhood of $\xi = \xi_c$ (cf. Rayleigh 1945, § 369; Lin 1955, (8.1.8)).

We shall attempt here only an approximate determination of β through (4.6), using the approximation

$$\phi = w(\xi)e^{-\xi}. \tag{5.1}$$

It is evident that (5.1) satisfies the boundary conditions (3.8 a, b), while the differential equation (3.6) is satisfied if either $|w''| \gg |w|$ (small ξ) or $|w''| \ll |w|$ (large ξ). These conditions are not, of course, sufficient to guarantee that the approximation (5.1) is uniformly valid for all ξ; in particular, it fails completely at the singular point ξ_c. We anticipate, nevertheless, that, in conjunction with (4.3) and (4.6), it will lead to an approximation to the magnitude of ϕ_c which is adequate for at least the qualitative description of the spectral distribution of the parameter β. We shall be content with this goal, remarking that the difficulties associated with the accurate determination of wind profiles scarcely justify anything more elaborate at this time.

Substituting (5.1) in (4.6) and that, in turn, in (4.3), we obtain,

$$\beta = -\pi \frac{w_c''}{w_c'^3}\left[\int_{\xi_c}^{\infty} e^{-\xi} w^2\, d\xi\right]^2. \tag{5.2}$$

We shall evaluate the integral in (5.2) for a logarithmic profile, which, for turbulent flow over water, has the support of both theory (see Coles (1956) for a recent survey) and experiment (Roll 1948; see also Hay 1955; Ellison 1956). The form that most directly fits the description of § 3 above is

$$U(y) = U_0 + U_1 \log(y/y_0), \tag{5.3 a}$$

where U_0 is the lower bound of the logarithmic profile and U_1 the reference velocity, which is now defined as the coefficient of the *natural* logarithm in the velocity profile. The majority of experimental results are presented in the form (except that common logarithms are apt to be used)

$$U(y) = (U_*/\kappa)\log(y/z_0), \tag{5.3 b}$$

where $U_* = (\tau_0/\rho_a)^{1/2}$ is Prandtl's *shearing stress velocity*, τ_0 the shear stress at the surface, κ Kármán's universal *turbulence constant*, and z_0 an *effective roughness parameter*, in the notation of Roll (1948); the definitions of U_* and κ are those commonly adopted, but z_0 has been used in somewhat different senses by different writers. We remark that the reference pressure $\rho_a U_1^2$ now is proportional to the shearing stress τ_0.

The distribution of (5.3) evidently holds only for a limited range of U. We designate the wind speed at the outer edge of this range as U_∞; for our purposes, $U_\infty = 10U_1 = 25U_*$ is a rough but adequate approximation for the wind speed at two metres above the water*. The determination of the lower limit of validity, U_0, is both more important and more uncertain, reflecting the fact that, in reality, there is not a sharp transition from

* The reference position of two metres is that adopted by Roll; at 10 metres $U_\infty = 30U_*$ might be a better estimate.

logarithmic profile to sub-layer (laminar or otherwise). Estimates of both U_0 and z_0 will be discussed in § 7.

Substituting (5.3 a) or (5.3 b) in (3.5 b) and (5.2) and noting that, by definition,

$$c = U_0 + U_1 \log(y_c/y_0) = U_1 \log(y_c/z_0), \tag{5.4 a, b}$$

we obtain

$$w = \log(\xi/\xi_c), \tag{5.5}$$

$$\beta = \pi\xi_c \left[\int_{\xi_c}^{\infty} e^{-\xi} \log^2(\xi/\xi_c)\, d\xi \right]^2 \tag{5.6 a}$$

$$= \pi\xi_c^3 \left[\int_1^{\infty} e^{-\xi_c u} \log^2 u\, du \right]^2. \tag{5.6 b}$$

The integral in (5.6 b) may be evaluated for small or moderate values of the argument ξ_c by writing the range of integration $(1, \infty)$ as the difference $(0, \infty)$–$(0, 1)$, identifying the infinite integral as the Laplace transform of $\log^2 u$ [Erdélyi 1954, § 4.6 (13)], and evaluating the finite

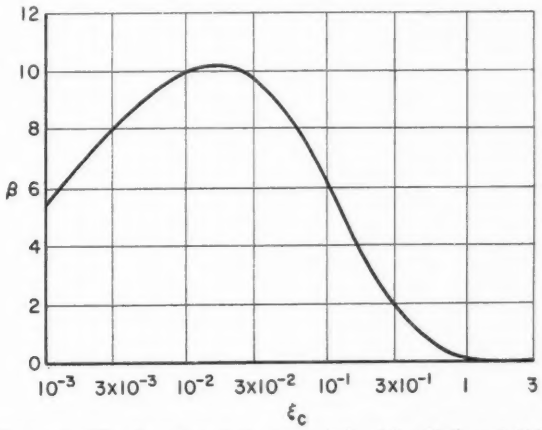

Figure 1. The function $\beta\,(ky_c)$, as defined by (5.6) and (5.7)

integral as a power series, following the expansion of the exponential. An asymptotic development for large ξ_c may be obtained by the usual device of repeated integration by parts or by expanding $\log^2 u$ in powers of $(u-1)$. The end results of these operations are

$$\beta(\xi_c) = \pi\xi_c \left\{ \tfrac{1}{6}\pi^2 + \log^2(\gamma\xi_c) + 2 \sum_1^{\infty} \frac{(-)^n \xi_c^n}{n!\, n^2} \right\}^2, \tag{5.7 a}$$

$$\beta(\xi_c) \sim 4\pi\xi_c^{-3} e^{-2\xi_c} (1 - 6\xi_c^{-1} + 31\xi_c^{-2} \ldots), \tag{5.7 b}$$

where $\log\gamma \doteqdot 0{\cdot}5772$ is Euler's constant. We find that β has a maximum value of 10·2 at $\xi_c = 0{\cdot}017$ and that (5.7 a) is adequate for numerical calculation over the range of practical interest ($\xi_c < 1$). The asymptotic series (5.7 b), on the other hand, is accurate only for very large ξ_c and is of interest primarily as an indication of the exceedingly sharp cut-off of the damping ratio for very large values of ky_c.

The numerical values of β are plotted in figure 1.

6. Damping ratio *vs* wave speed

We now proceed to express β and ζ_a as functions of the dimensionless wave speed c/U_1. The corresponding wave number for gravity waves may be determined from (2.8 a), whence

$$k = g/c^2. \tag{6.1}$$

Solving (5.4 b) for y_c and multiplying by (6.1) then yields

$$\xi_c = ky_c = \Omega(U_1/c)^2 e^{c/U_1}, \tag{6.2}$$

where we have introduced the dimensionless, wind-profile parameter

$$\Omega = gz_0/U_1^2. \tag{6.3}$$

The function ky_c/Ω is plotted in figure 2. The corresponding values of β, as calculated from (5.7 a), and ζ_a/s, as given by (2.10), namely,

$$\zeta_a/s = (U_1/c)^2\beta[\Omega(U_1/c)^2 e^{c/U_1}], \tag{6.4}$$

are plotted in figures 3 and 4 for values of Ω that appear to be typical (cf. § 7).

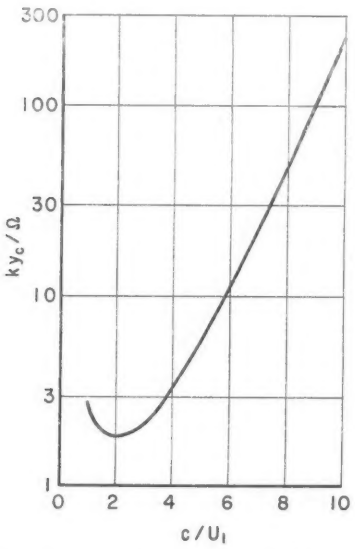

Figure 2. The variation of ky_c with wave speed; see (6.2) and (6.3)

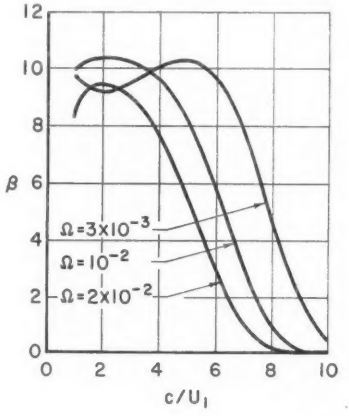

Figure 3. The parameter β, as determined by (5.7 a) and (6.2); cf. figures 1 and 2.

The damping ratio associated with viscous action in the water, based on the assumptions stated in § 1 (free surface with laminar flow), is given by (Lamb 1945, § 348)

$$\zeta_w = -4\nu_w k/c. \tag{6.5}$$

Eliminating k through (6.1) and superimposing ζ_a and ζ_w, we obtain

$$\zeta = \beta s U_1^2 c^{-2} - 4\nu_w g c^{-3}. \tag{6.6}$$

Equating this expression to zero yields

$$U_1 = (4\nu_w g/s)^{1/3}[(c/U_1)\beta]^{-1/3} \tag{6.7 a}$$

for the wind speed at which the energy available from the air stream is just sufficient to maintain a small amplitude gravity wave of speed c against laminar dissipation in the water. If we take the c.g.s. numerical values $\nu_w = 10^{-2}$, $g = 980$, and $s = 1{\cdot}2 \times 10^{-3}$, (6.7 a) reduces to

$$U_1 = 32[(c/U_1)\beta]^{-1/3}\,\text{cm/sec} \quad \text{for } \zeta = 0. \tag{6.7 b}$$

This result is plotted in figure 5 using the values of β given in figure 3.

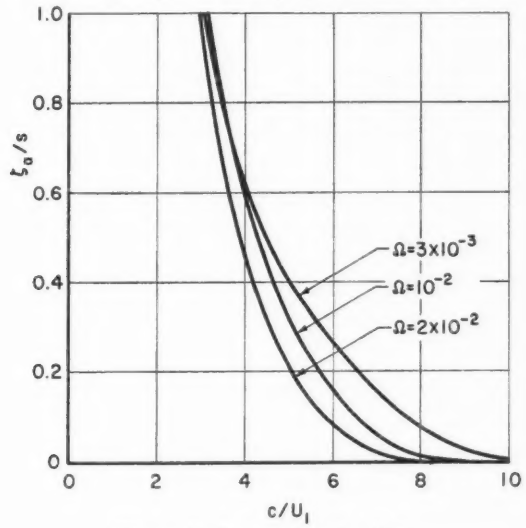

Figure 4. $\zeta_a s$, as given by (6.4).

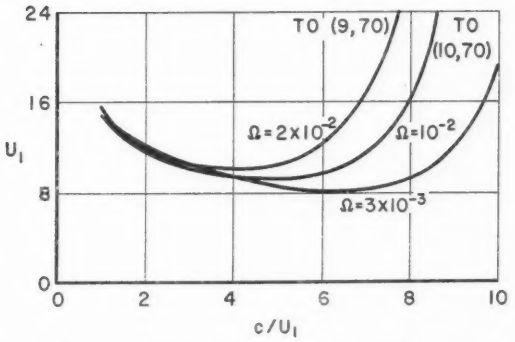

Figure 5. The minimum reference speed U_1 for the initiation of gravity waves, as given by (6.7 b); the wind speed at two metres is roughly $10U_1$.

It appears from the results of figure 5 that the minimum wind speed (at two metres) for the initiation of *gravity* waves is of the order of magnitude of 100 cm/sec and might be as small as 80 cm/sec; the corresponding wave speeds would lie somewhere between 40–50 cm/sec. These figures are close

to the respective measurements of Jeffreys (1924) and Scott Russell (1844), but it is important not to lose sight of the approximations on which they are based. The equivalent value of Jeffreys' sheltering coefficient, as given by

$$\beta_J = \beta U_1^2/(U_\infty - c)^2, \tag{6.8}$$

is roughly 0·3 (for $\Omega = 10^{-2}$, $c = 5U_1$, $U_\infty = 10U_1$).

7. The wind profile parameters

The available measurements on wind profiles over water (see Roll 1948; Neumann 1948; Hay 1955; also other papers cited by Ursell 1956 and Ellison 1956) all tend to confirm the logarithmic law posed in (5.3), but they provide little information on the laminar sub-layer and conflicting data on the effective roughness parameter. We consider here three possibilities.

(*a*) *Aerodynamically smooth flow*

If the departures from a plane surface do not penetrate the laminar sub-layer and the neglect of surface currents is justified the velocity profile should resemble that for flow near a smooth wall. Nikuradse's data yields (Prandtl 1952, p. 126)

$$U(y)/U_* = 5{\cdot}75\log_{10}(U_* y/\nu_a) + 5{\cdot}50, \tag{7.1}$$

corresponding to $\kappa = 0{\cdot}40$ and $z_0 = \nu_a/9{\cdot}06U_*$ in (5.3 b). We then have

$$z_0 = 1{\cdot}7\times 10^{-2}U_*^{-1}, \qquad \Omega = 2{\cdot}7U_*^{-3}, \tag{7.2 a, b}$$

in c.g.s. units ($g = 980$, $\nu_a = 0{\cdot}154$).

Roll has made measurements for wind blowing over smooth ponds which yield values of z_0 which are not only appreciably smaller than those predicted by (7.2 a) for $U_* < 10$ cm/sec but also decrease with U_*. The scatter in these data for small U_* is considerable, but the trend is quite definite. The most plausible explanation would appear to be in the existence of a surface current in the water, but further investigation of this point is desirable*.

If the logarithmic profile of (7.1) is matched in ordinate and slope to a linear profile it is found that $U_0 = 7{\cdot}8U_*$ at $U_* y_0/\nu_a = 1/\kappa$ (cf. Roll 1948). An examination of pipe data (Prandtl 1952, p. 128) reveals that the logarithmic and linear profiles are established only for $U/U_* > 15$ and $U/U_* < 5$, respectively, but the logarithmic profile does not appreciably overestimate U''/U' as a function of U for $U > 8U_*$.

(*b*) *Moderate wind over wavy water*

Roll found that for values of U_* between about 10 and 30 cm/sec and heights up to two metres above a disturbed surface his measurements (which exhibited a much smaller scatter than those for $U_* < 10$ cm/sec)

* Prandtl (1952, p. 130), referring to this data, states that " for values of U_* below about 10 cm/sec the flow in the lowest 2 metres seems to be laminar ". This interpretation is difficult to accept, since his own " universal velocity function " (*ibid*, p. 128) implies a value of y from 1–3 mm for transition at $U_* = 10$ cm/sec.

could be fitted to (7.1) when y was measured from the wave crests. In terms of the distance from the mean surface, he found

$$U/U_* = 5{\cdot}75 \log_{10}[U_*(y+a)/\nu_a] + 1{\cdot}85, \tag{7.3}$$

where a was the amplitude of the surface waves. This implies that

$$z_0 = 7{\cdot}3 \times 10^{-2} U_*^{-1}, \qquad \Omega = 11{\cdot}6 U_*^{-3}. \tag{7.4 a, b}$$

It should be recalled that we have defined y as a streamline parameter, so that the velocity profile of (7.3) is not directly significant and serves only to illustrate the magnitude of possible deviations from (7.1). We shall see (figure 6) that the resulting differences in ζ_a are negligible.

The results (7.3) and (7.4) are characteristic of aerodynamically smooth flow, so that it appears reasonable again to assume that (7.3) is valid down to about $U_0 = 8U_*$.

(*c*) *Fully developed rough flow*

It has been argued (cf. Ursell's (1956) and Ellison's (1956) remarks) from dimensional considerations that, since (i) the apparent roughness of the water surface depends on the waves produced by the wind and (ii) the roughness parameter in pipe flows is found to be independent of viscosity and to depend only on the nature of the roughness for $U_* z_0/\nu_a > 3$ (*rough flow*), z_0 should be proportional to U_*^2/g for sufficiently rough flow. We remark here that this conclusion appears to assume that capillary ripples are unimportant in determining the apparent roughness, although this would appear doubtful; nevertheless, Hay (1955) has made measurements for values of U_* between about 25 and 55 cm/sec that tend to confirm this hypothesis and lead to the approximate results (which easily might be off by as much as 30%; cf. Ellisons' remarks)

$$z_0 = 8 \times 10^{-5} U_*^2, \qquad \Omega = 1{\cdot}25 \times 10^{-2}. \tag{7.5 a, b}$$

The corresponding value of U_0 is probably best estimated as that at $y_0 = 30 z_0$, namely $U_0 = 8{\cdot}5 U_*$.

The discrepancy between (7.4) and (7.5) emphasizes the order of magnitude of the uncertainties with which we deal; for example, at $U_* = 30$ cm/sec, where the data of Roll and Hay overlap, the values of z_0 obtained from (7.4 a) and (7.5 a) are 2×10^{-3} cm and 7×10^{-2} cm, respectively. In any event, our model cannot be expected to have more than qualitative significance for rough flow.

The foregoing results are illustrated in figures 6 and 7, where we have plotted ζ_a *vs* λ $(= 2\pi/k)$ for the combinations of data indicated in table 1 (the values of Ω used in the actual calculations were rounded off to the parenthetic values in order to reduce the numerical work; the rounding errors are smaller than the other uncertainties). The differences in z_0 for given U_* are seen to be relatively unimportant. We again emphasize that the results for the higher wind speeds must be regarded as especially dubious in consequence of the linearization. The value of $-\zeta_w$ given by (6.5) also is plotted as a reference base, but the actual damping from the

water almost certainly would be very much larger for values of U_* as high as 30 cm/sec. The values of $c/U_1 = 2, 3, 4$, and 5, corresponding to various, possible lower limits $U_0 = c$ are marked off; on the basis of the preceding

Item	U_* (cm/sec)	z_0 (cm)	U_{200} (cm/sec)	Ω	Eqs.
a_5	5	$3{\cdot}4\times10^{-3}$	110	$2{\cdot}2\times10^{-2}$ (2×10^{-2})	(7.1), (7.2)
a_{10}	10	$1{\cdot}7\times10^{-3}$	290	$2{\cdot}7\times10^{-3}$ (3×10^{-3})	(7.1), (7.2)
b_{10}	10	$7{\cdot}3\times10^{-3}$	250	$1{\cdot}2\times10^{-2}$ (10^{-2})	(7.3), (7.4)
b_{30}	30	$2{\cdot}4\times10^{-3}$	850	$4{\cdot}3\times10^{-3}$ (3×10^{-3})	(7.3), (7.4)
c_{30}	30	$7{\cdot}2\times10^{-2}$	600	$1{\cdot}2\times10^{-2}$ (10^{-2})	(7.5)

Table 1. The parenthetic values for Ω were used in calculating figure 6.

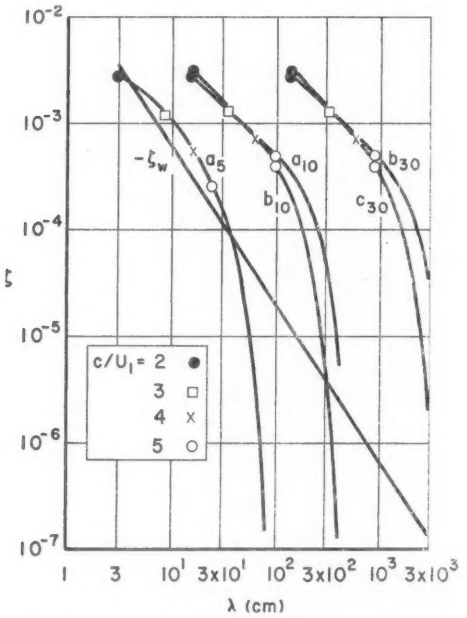

Figure 6. Damping *vs* wavelength for the data of table 1.

remarks, the logarithmic profile probably does not overestimate ζ_a for $c > 4U_1 (= 10U_*)$ but certainly would do so for values of c as low as $2U_1$ $(= 5U_*)$.

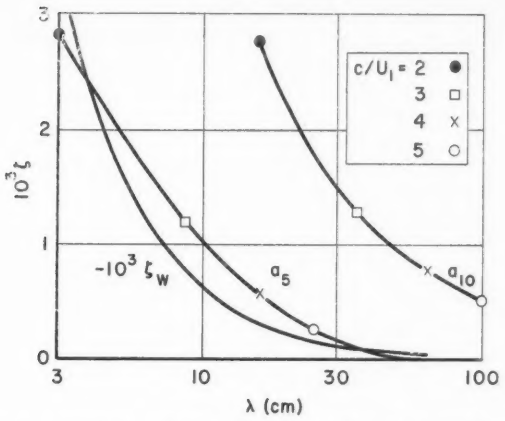

Figure 7. Linear plot of cases a_5 and a_{10}; cf. table 1 and figure 6.

8. Mean tangential wave drag

A rough check on the mean value of β (the minimum wind speed estimates of § 6 provide a check only on values of β in the neighbourhood of its maximum) may be obtained by calculating the (space or time) mean tangential drag on the waves, namely,

$$\tau_x = \overline{\mathscr{R}\{p_a(ik\eta)^*\}}. \tag{8.1}$$

Munk (1955), using data from Van Dorn's (1953) experiments, has obtained the expression

$$\tau_x = c'\rho_a(g\nu_w)^{-1/3}U_\infty^3, \tag{8.2}$$

where $c' = 0{\cdot}68\times10^{-6}$ if U_∞ is measured at 10 metres above the surface.

Let $S(k,\theta)$ be the power spectral density of the surface* in the notation of Eckart (1953 a), θ being the angle between wavefront normal and wind direction (so that the effective component of the wind is $U\cos\theta$); the power spectral density of the wave slope then is $k^2S(k,\theta)$, and substituting p_a in (8.1) from (2.3) yields

$$\tau_x = \rho_a U_1^2 \int_0^\infty \int_{-\pi}^{\pi} \beta\cos^2\theta k^2 Sk\; dkd\theta. \tag{8.3}$$

The spectrum for a fully developed sea (Neumann, as cited by Munk (1955)) may be approximated by

$$\left.\begin{aligned} k^2S(k,\theta)k\; dkd\theta &= 4(2\pi)^{-1/2}\theta_0^{-1}\sigma^2 e^{-2(c/U_\infty)^2}d(c/U_\infty)\,d\theta(|\theta|<\tfrac{1}{2}\theta_0),\\ &= 0 \quad (|\theta|>\tfrac{1}{2}\theta_0), \end{aligned}\right\} \tag{8.4}$$

where σ^2 is the mean square slope, which may be expressed in the form

$$\sigma^2 = b(g\nu_w)^{-1/3}U_\infty \tag{8.5}$$

* The surface displacement now is assumed to be composed of a distribution of sine waves having random phases.

with $b = 1{\cdot}1 \times 10^{-4}$ (Cox & Munk 1954). Munk (1955) also introduces a sheltering coefficient β_M (s in his notation), defined such that

$$c' = \beta_M \, bf(\theta_0), \tag{8.6}$$

where, for our model (in which $U\cos\theta$ would appear as the effective wind speed), $f(\theta_0)$ is simply the mean value of $\cos^2\theta$ over $(-\frac{1}{2}\theta_0, \frac{1}{2}\theta_0)$. Substituting (8.4) and (8.5) in (8.3) and comparing the result with (8.2) and (8.6), we find

$$\beta_M = \frac{4}{\sqrt{(2\pi)}}\left(\frac{U_1}{U_\infty}\right)^3 \int_0^{U_\infty/U_1} \beta e^{-2(c/U_\infty)^2}\, d\left(\frac{c}{U_1}\right). \tag{8.7}$$

Using the values of β given by figure 3 with $\Omega = 10^{-2}$ (the value most likely to be representative of a fully developed sea), assuming $\beta \equiv 0$ for $c < 3U_1$, and choosing $U_\infty = 12U_1$ (at 10 metres above the surface), we obtain $\beta_M = 2{\cdot}0 \times 10^{-2}$, which compares with the value 10^{-2} inferred by Munk from observed rates of wave growth. The corresponding value of c', using $b = 1{\cdot}1 \times 10^{-4}$, is $2{\cdot}1 \times 10^{-6}\ \overline{\cos^2\theta}$; with Munk's value of θ_0, namely 130°, we have $c' = 1{\cdot}5 \times 10^{-6}$, which compares with the experimental value of $0{\cdot}68 \times 10^{-6}$. We emphasize, however, that our estimate of β_M might be off by a factor of two either way in consequence of our estimates of Ω and c_0.

We conclude that the calculated value of c' is in agreement with the experimental value in at least order of magnitude, from which we infer that the mean value of β, weighted as in (8.7), also is of the right order of magnitude.

9. Conclusions

We conclude that the model of an inviscid shear flow in air over water having zero mean motion provides an explanation of the generation of gravity waves which is in qualitative agreement with observations of minimum wind speed and mean tangential drag. Quantitative comparisons would require, on the theoretical side, a more accurate solution of the boundary value problem, and, on the experimental side, much more precise data on wind profiles. We also emphasize that the model is intrinsically limited by the neglect of turbulent interaction between surface wave and wind profile and the linearization of the equations of motion. Finally, we note that the results of § 2–5, regarding energy transfer from shear flows to surface waves, are applicable to such problems as the flutter of membranes and panels.

I am grateful to Lester Lees of the California Institute of Technology, Nicholas Rott of Cornell University, Cark Eckart, Walter Munk and Charles Cox of the Scripps Institute of Oceanography, and my colleague Kurt Forster, for helpful discussion and criticism during the course of the above research.

Appendix

The Euler and continuity equations for a viscous incompressible fluid can be written

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial p_{ij}}{\partial x_j}, \tag{A1 a}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{A1 b}$$

where x_i denotes a Cartesian coordinate, u_i a velocity component, p_{ij} a stress tensor component, ρ the fluid density, and the usual summation convention is implied. We resolve the dependent variables according to

$$u_i = U_i + u_i' + u_i'', \qquad p_{ij} = P_{ij} + p_{ij}' + p_{ij}'', \tag{A2 a, b}$$

where $U_i + u_i'$ and $P_{ij} + p_{ij}'$ represent a solution to (A1 a, b) having the two-dimensional (x_1 and x_2) mean values (with respect to either x_3 or t) U_i and P_{ij} plus turbulent fluctuations u_i' and p_{ij}', and u_i'' and p_{ij}'' represent a small perturbation with respect to this solution. Substituting (A2 a, b) in (A1 a), neglecting terms of second order in the perturbation flow, and invoking the requirement that the unperturbed flow satisfy (A1 a), we obtain

$$\frac{\partial u_i''}{\partial t} + (U_j + u_j) \frac{\partial u_i''}{\partial x_j} + u_j'' \frac{\partial (U_i + u_i')}{\partial x_j} = \frac{1}{\rho} \frac{\partial p_{ij}''}{\partial x_j}, \tag{A3 a}$$

$$\frac{\partial u_i''}{\partial x_i} = 0. \tag{A3 b}$$

Taking mean values with respect to x_3, we may place the results in the form

$$\frac{\partial \overline{u_i''}}{\partial t} + U_j \frac{\partial \overline{u_i''}}{\partial x_j} + \overline{u_j''} \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} (\overline{p_{ij}''} - \overline{r_{ij}''}), \tag{A4 a}$$

$$\frac{\partial \overline{u_i''}}{\partial x_i} = 0, \tag{A4 b}$$

where we have introduced the *perturbation Reynolds stress* (invoking the equations of continuity for both u_i' and u_i'' in its derivation)

$$\overline{r_{ij}''} = \rho(\overline{u_i' u_j''} + \overline{u_j' u_i''}) \tag{A5 a}$$

$$= \rho[\overline{u_i'(u_j'' - \overline{u_j''})} + \overline{u_j'(u_i'' - \overline{u_i''})}], \tag{A5 b}$$

with (A5 b) following from (A5 a) by virtue of $\overline{u_i'} = 0$.

Equations (A4 a, b) are the equations of motion governing the two-dimensional perturbation of a two-dimensional shear flow if *two-dimensional* is understood to imply *mean value with respect to the transverse coordinate.* They differ from the equations of motion for a non-turbulent perturbation flow only in the presence of the $\overline{r_{ij}''}$, which represent the interaction between the fluctuations in the original and perturbation flows and, as might have been anticipated, are the first order perturbations of the usual Reynolds stresses $\rho \overline{u_i' u_j'}$.

Equations (3.1 a, b, c) follow from (A4 a, b) if we take $x_1 = x$, $x_2 = y$, $U_1 = U(y)$, $U_2 = U_3 = 0$, $\overline{u_1''} = u$, $\overline{u_2''} = v$, $\overline{p_{ij}''} = -\delta_{ij} p$, and $\overline{r_{ij}} = 0$.

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Note added at proof stage

A stochastic analysis recently presented by Phillips (1957) develops a model which appears more realistic than that proposed by Eckart (1936 b) and arrives at results in reasonable agreement with experiment. It seems that the mechanism proposed by Phillips is complementary to that presented here, but further investigation is required to establish the appropriate connections.

On the fine structure of turbulent flows

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SUMMARY

The turbulence produced by a multiplicity of small air jets has been investigated and comparisons are made with other turbulent flows. The eddy Reynolds number is large ($R_\lambda = 250$).

The energy spectrum was measured, as well as the skewness and kurtosis of $\partial u/\partial x$. The effect of the finite length of the hot wire is considered and the corrected results indicate that the spectrum follows the law of Kolmogoroff in an intermediate range and appears to fall off with the (-6)-power of the frequency in the viscous range. This range is limited at the upper end of the frequency range by electrical noise. Special precautions reduced this noise to the level of thermal agitation in the hot wire.

The ultimate limit to spectral analysis is imposed by the molecular agitation of the gas. This limit is evaluated and compared with the spectrum of turbulence. It appears that the spectrum of $\partial^3 u/\partial x^3$ merges with the spectrum of molecular agitation without a distinctive separation.

1. INTRODUCTION

The fine structure of turbulent flows has been the object of numerous investigations but complete understanding has not yet been reached. In this paper, we describe certain recent results that may contribute to the clarification of the situation.

The theory of turbulence becomes greatly simplified if one assumes incompressibility, homogeneity and isotropy of the velocity field. Under such conditions, one can define a one-dimensional spectrum $F(k)$ which is a function of a wave-number k. Let us define k_E as the wave-number such that half the kinetic energy of the motion is contributed by the region $k > k_E$ and let us define k_ν as the wave-number such that half of the viscous energy dissipation occurs in the region $k > k_\nu$.

Two important theoretical predictions about $F(k)$ are as follows (Batchelor 1953, ch. 6). According to Kolmogoroff, the spectrum should fall as $k^{-5/3}$ in some subrange of wave-numbers in the range $k_E < k < k_\nu$, provided that $k_E \ll k_\nu$. According to Heisenberg, the spectrum should fall as k^{-7} in the region $k > k_\nu$. These two statements can be formulated in a single expression, and the theory shows that $F(k)$ should cross a limit given by $c\nu^2 k$ in the vicinity of $k = k_\nu$, where c is a universal constant and ν the kinematic viscosity. Physically, this means that spectral components such that $F < c\nu^2 k$ are strongly influenced by viscosity.

The theory of Heisenberg implies that the third derivative of the velocity does not exist, or, more exactly, that the mean-square of this quantity, given by the integral of $k^6F(k)$, does not converge.

From Kolmogoroff's theory it can also be deduced that the parameter S (skewness of $\partial u/\partial x$) should be a constant, provided $k_E \ll k_\nu$.

The experimental studies of the fine structure of turbulence have been carried out in a single gas (air), virtually at a single temperature and pressure. The velocity fluctuations are sensed by hot-wire anemometers and the effect of the finite length of the wire is often disregarded. The hot-wire anemometer gives an electric signal proportional to the velocity fluctuation plus a certain amount of electronic noise, so that some kind of filter is desirable to reject unnecessary noise.

In general, the turbulent fluid passes by the hot-wire with a mean velocity U which is much larger than the velocity fluctuation and, with the hypothesis made by Taylor, the time derivative of the signal is taken as a fair approximation to the space derivative. This allows the measurement of quantities such as:

$$u' = \langle u^2 \rangle^{1/2}, \qquad \lambda = u' \langle (\partial u/\partial x)^2 \rangle^{-1/2},$$

where $\langle\ \rangle$ denotes a time average, u is the velocity fluctuation in the direction of the x-axis and λ is the dissipation length parameter. If $k_E \ll k_\nu$, it can be shown that $\lambda^{-3} \doteqdot k_E k_\nu^2$, and the eddy Reynolds number

$$R_\lambda = u'\lambda/\nu$$

becomes proportional to $(k_\nu/k_E)^{2/3}$. The verification of the theoretical laws thus requires a large value of R_λ, say $R_\lambda \geqslant 100$.

Grid flows

A large variety of investigations have been carried out in the turbulent wake of a plane grid obstructing a parallel flow. The requirements of homogeneity and isotropy are reasonably well satisfied here and for low velocities incompressibility is a reasonable approximation. R_λ usually varies from 20 to 100. In practice, it becomes difficult to observe the fine details of the motion when $R_\lambda > 60$. This is due to the fact that $u'/U \leqslant 0{\cdot}02$, resulting in a weak electric signal from the hot-wire. At some frequency ω_N the spectrum of the signal merges into the noise of the instruments, consequently the smallest observable details of the motion are comparable to the length $l_N = U/2\pi\omega_N$. With a given grid size and a given hot-wire, R_λ increases as $U^{1/2}$ whereas l_N increases at least as U. Therefore, as R_λ increases, the resolution of the fine structure deteriorates. To avoid this effect, one should use lower viscosities or larger wind tunnels. Because of these limitations, it was not possible either to really support or to disprove the theoretical spectral laws with the work on grid flow.

It has been found that S varies very little and that quantities such as $\langle(\partial u/\partial x)^2\rangle$ and $\langle(\partial^2 u/\partial x^2)^2\rangle$ are related to each other in the way required by the Navier–Stokes equations. Measurements of the quantity $\langle(\partial^3 u/\partial x^3)^2\rangle$ have been reported by Batchelor & Townsend (1949), Townsend (1951),

and Stewart & Townsend (1951); Liepmann, Laufer & Liepmann (1951) carried out similar investigations of the third derivative but report more cautiously that "it appears to be finite".

Townsend (1951) proposed a simple model for the fine structure of turbulence based on stretching of vortex sheets or vortex lines. It leads to a spectrum falling as an exponential function of k^2 when k is large. Measurements in grid flows in the range $15 \leqslant R_\lambda \leqslant 30$ are in good agreement with the model, but discrepancies seem to appear as R_λ increases.

Shear flows

The structure of turbulent shear flows has been studied by Laufer (1950, 1953), Klebanoff & Diehl (1951), Corrsin & Uberoi (1951) and others. The departure from isotropy and homogeneity are serious but it affects principally the spectral components with wave-numbers near k_E. In a flow with $R_\lambda = 300$, Laufer (1950) found that the fine structure is nearly isotropic, that a portion of the spectrum falls as $k^{-5/3}$ and that the high frequency end of the spectrum falls as k^{-7}. (He obtained this without applying a correction for the finite length effect of the hot-wire.) This last result implies that $\langle(\partial^3 u/\partial x^3)^2\rangle$ is infinite or at any rate that it is determined by values of k too large to be measured.

In an attempt to clarify this situation, we designed a simple device to produce turbulence with large R_λ and small l_N. The object of this paper is to describe the procedure and to comment on the results.

2. Experimental installation

The 'Porcupine'

A plywood box $60 \times 60 \times 120$ cm has five perforated sides, as shown on figure 1. The sixth side is faired into a duct leading to a 20 h.p. blower.

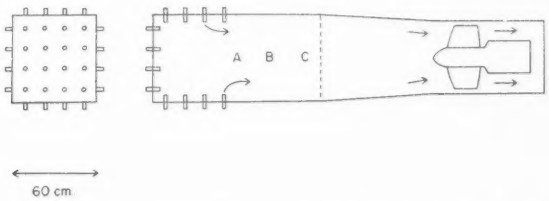

Figure 1. The 'Porcupine'. The mixing of 80 small jets produces a strong turbulence in the region marked *A*, *B*, *C*.

Each of the 80 openings is fitted with a round tube, 10 cm long and 2·5 cm inside diameter. In operation the air enters the box through these tubes, forming 80 jets which merge into a turbulent flow exhausted into the duct by the blower. The Reynolds number based on velocity and diameter of a single jet is about 35 000. The mean velocity of the flow in the parallel section is 450 cm/sec and the turbulence levels measured at the points *A*, *B*, *C* on the box axis are respectively 30%, 12% and 5%.

At point A, the turbulence is so strong that the hot-wire signal is not proportional to the velocity fluctuation, because of the intrinsic non-linearity of the instrument. At points B and C this distortion becomes less serious. The resolution length l_N is larger at B than at C by about 60%. and, as a compromise between the distortion and the loss of resolution, the author decided to perform all measurements at point B.

No systematic variations of U and u' over the plane normal to the mean flow were found, at distances up to 15 cm from the box axis. Both quantities varied by 5 to 15% between one run and another and it appears that the merging of the jets produces different flow patterns, as observed elsewhere by Corrsin (1944).

Hot-wire and preamplifiers

Pt wires (diameter 1·25 microns, length 1 mm) heated up to twice the cold resistance were used. Whenever it became essential to minimize the electric perturbations (60 cycles ripple or contact noise) each wire was heated with a separate 22 volts battery and a fixed wire-wound series resistor; all other metering devices were completely disconnected.

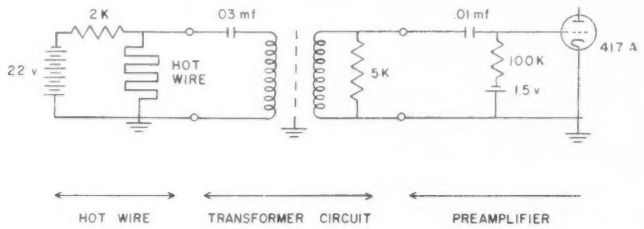

Figure 2. Input circuit for low electrical noise.

Two low noise preamplifiers were used, each with a Weston 417 A triode. The noise of each preamplifier is equivalent to that of a 600 ohms resistor applied to the input. With the amplifiers in parallel, a single channel is formed with a noise of 300 ohms. For measurements in the high frequency range, lower noise figures were achieved with the transformer input circuit of figure 2. The relatively small coupling condenser renders the output proportional to the time derivative of the signal, up to 20 kc/s. Above this frequency, the wire resistance and other impedances become significant and the overall response deteriorates. With a transformer ratio of 10 to 1, the input noise at 15 kc/s becomes equivalent to the thermal noise of a 50 ohms resistor at room temperature.

With an unheated wire of 60 ohms the total equivalent noise-generating resistance is 110 ohms. With the same wire heated to 120 ohms, that is, to approximately 600° K, we can expect a noise of about 300 ohms. This means that the amplitude of the noise should increase by about 260%, when the heating current is turned on. (This increase is noticeable on figure 5, at $k > 300$, in the range where the electronic noise dominates the turbulent signal.)

We verified that the battery did not make an additional contribution to the noise. For this purpose we replaced the hot-wire with a 200 ohms wire-wound resistor, and observed that the passage of the heating current did not affect the noise of the wire-wound resistor.

Thermal lag compensation

The thermal inertia of the hot-wire appreciably attenuates the signal above 600 c/s. With the transformer input of figure 2, the coupling condenser provides the desired compensation from 600 c/s to 20 kc/s. It would be inadvisable to modify this circuit and obtain a constant gain between 3 and 600 c/s. Indeed, the low frequency components of the signal would be so large that they would saturate the preamplifier.

With a capacitive coupling into the preamplifier, it was necessary to limit the response of the preamplifier to the range of 200 c/s to 20 kc/s, in order to avoid this saturation by low frequency components. When the transformer was not in use, the thermal compensation was provided by an operational amplifier with an appropriate feed-back loop. However, the presence of a large turbulent signal made it difficult to adjust the time constant of the circuit by the usual square wave method.

For measurements of u' or of the low frequency portion of the spectrum, the preamplifier was by-passed completely.

Differentiation and filtering

An operational amplifier was used in the manner of analogue computing. The response of each differentiating unit increased with ω up to 20 kc/s and decreased with ω^{-1} from 40 kc/s upward. This provided the appropriate filtering of the high frequency components of the electronic noise.

Wave analysers

Between 3 and 7500 c/s a General Radio wave analyser with a band-width proportional to the tuned frequency was used, and, from 100 c/s to 15 kc/s, a Hewlett Packard analyser built around a modulator and four amplifiers tuned for 20 kc/s. In order to avoid the formation of modulation images, it was necessary to insert ahead of this wave analyser a low-pass filter with cut-off at 15 kc/s. From 7 kc/s to 60 kc/s a Sierra analyser with constant band-width was used.

The output of each analyser was squared, averaged, and reduced to a constant band-width, and the responses were matched at 100 c/s and 10 kc/s. Good agreement was found in the two regions where analysers overlap.

When the spectrum of the signal falls very rapidly, an analyser may be unable to separate the components properly. Indeed a tuned amplifier driven with a frequency lower than the resonance frequency has a gain proportional to the frequency. With four tuned amplifiers and a signal whose spectrum falls as ω^{-8}, the output of the analyser is seriously in error. This difficulty can be avoided if the signal is differentiated once or twice

before reaching the analyser, since each differentiation multiplies the spectrum by ω^2.

3. The energy spectrum (without correction for wire length)

With a wave analyser tuned on frequency ω the corresponding wave-number is $k = 2\pi\omega/U$ and the one-dimensional spectrum $F(k)$ is equal to the mean-square of the analyser output multiplied by a constant such that

$$\langle u^2 \rangle = \int_0^\infty F(k)\, dk.$$

In figure 3 we show $F(k)$ as a function of k, with linear scales. A check with a variable low-pass filter indicated that half of the mean-square of u is contributed by components below $k = 0{\cdot}1$ cm^{-1} and this confirms the existence of a maximum of F. It also suggests an energy scale of the order of 10 cm, which is comparable with the size of the box. A check with two crossed wires indicated that $u'/v' = 1{\cdot}3$, where v' refers to a velocity component normal to the mean flow. This represents an appreciable deviation from isotropy.

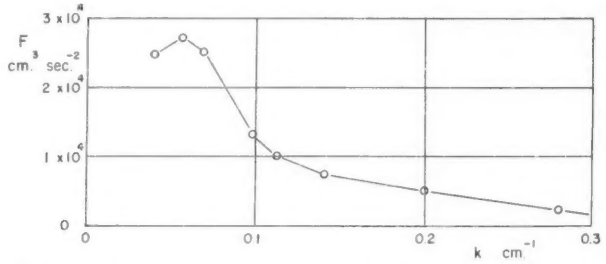

Figure 3. Energy spectrum *vs* wave-number with linear scales.

In figure 4 we show the spectrum with logarithmic scales. It follows the law of Kolmogoroff in the range $0{\cdot}15 \leqslant k \leqslant 15$. In the range $70 \leqslant k \leqslant 200$ it appears to follow the law of Heisenberg. For $k > 200$ the curve rises, but this is due to the electronic noise. These results have yet to be corrected for wire length effect; this will be done in the next section.

As previously discussed, it was suspected that the power law with which the spectrum falls off could be affected by the analyser itself. As a controlling experiment we measured the spectrum of $\partial^2 u/\partial t^2$, which is proportional to k^4F. The results (see figure 5) show a corresponding portion of the curve falling as k^{-3}, in agreement with the measurements of $F(k)$ (for a wire of finite length).

Figure 5 shows the spectrum of the noise, measured with a cold wire. At high frequencies it remains slightly below the signal obtained with the heated wire, as expected.

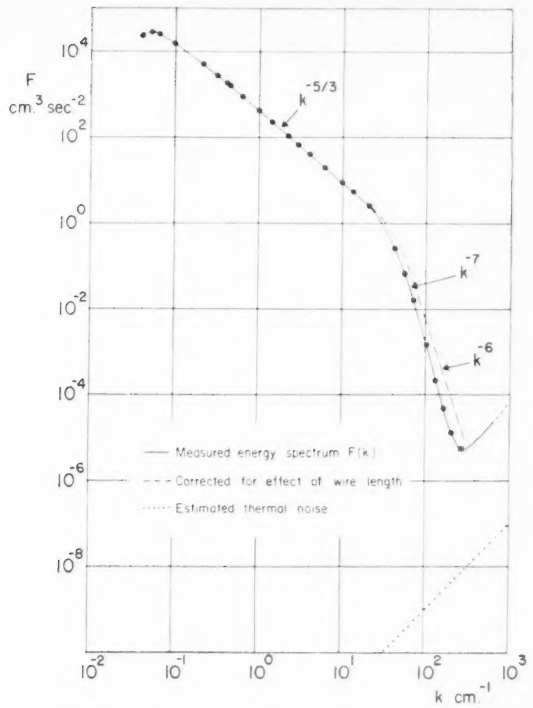

Figure 4. Energy spectrum *vs* wave-number with logarithmic scales.

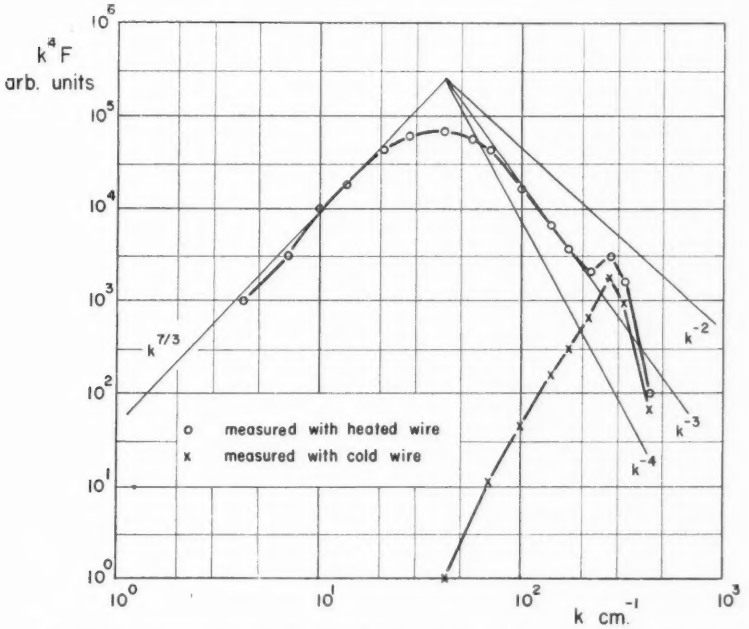

Figure 5. The spectrum of $\partial^2 u/\partial x^2$ (not corrected for wire-length effect).

In figure 6, the spectrum of figure 4 is compared with other results obtained in grid flows, in shear flows or in atmospheric turbulence. (It was necessary to convert the data published by several authors to the scales used here; this could not be done for the curves 7 and 8 and they have been given arbitrary translations.)

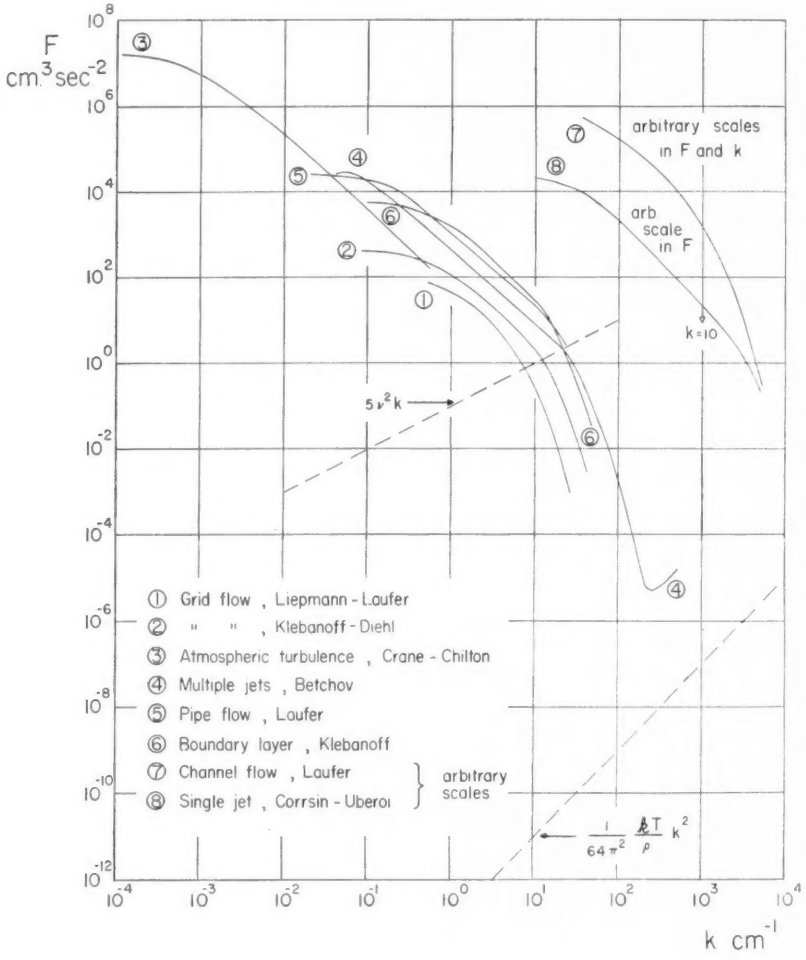
F
cm³sec⁻²
arbitrary scales in F and k
arb scale in F
k=10
5ν²k
① Grid flow , Liepmann - Laufer
② " " , Klebanoff - Diehl
③ Atmospheric turbulence , Crane - Chilton
④ Multiple jets , Betchov
⑤ Pipe flow , Laufer
⑥ Boundary layer , Klebanoff
⑦ Channel flow , Laufer
⑧ Single jet , Corrsin - Uberoi
arbitrary scales
k cm⁻¹

Figure 6. Comparison of several spectra.

We indicate by a dotted line the limit $c\nu^2 k$ with the arbitrary choice $c = 5$. Below this line, the spectrum is controlled by viscosity (see § 1).

The grid flows nos. 1 and 2 (Liepmann, Laufer & Liepmann 1951; Klebanoff & Diehl 1951) appear to have energy scales too small for a

well-developed range of Kolmogoroff's law. The shear flows nos. 5, 6, 7 (Laufer 1950, 1953; Klebanoff & Diehl 1951) and the author's turbulent flow no. 4 have such ranges. The curves 4, 5 and 6 seem to come together near $k = 20$ cm^{-1}. As for atmospheric turbulence, no. 3 (Crane & Chilton 1956), the measurements have not been extended to the fine scale. (A pioneering contribution by Gödecke (1935) in the range $0{\cdot}1 \leqslant k \leqslant 10$ unfortunately does not yield the spectrum.) It would be interesting and not too difficult to study the fine structure of atmospheric turbulence; it might reveal a viscous cut-off somewhere beyond $k = 10$.

Beyond the viscous limit, all the spectra fall rapidly enough to ensure the existence of $\langle(\partial u/\partial x)^2\rangle$ and $\langle(\partial^2 u/\partial x^2)^2\rangle$. As for $\langle(\partial^3 u/\partial x^3)^2\rangle$, it appears to depend upon spectral components beyond the range of present measurements.

4. Correction for wire length

The finite length of the hot-wire introduces a particular type of error in the measurement of spectra. A complete theory of this effect has been given by Kovásznay & Uberoi (1953), with the assumption that all portions of the wire are equally sensitive. In the case of the present investigation, the 'cold ends' are about 0·05 mm wide each and the wire length is 1 mm, and, according to Betchov (1948), this means that a correction for finite length is necessary for $k > 20$.

In figure 4 a dotted line shows the approximate values of the corrected spectrum. Furthermore, the theory of the wire-length effect shows that, if the measured spectrum falls as k^{-7}, the true spectrum should fall as k^{-6}. Since this correction has not been applied to the results of figure 5, the apparent seventh power law must be interpreted as evidence that the true spectrum falls off as k^{-6}. This means that the law proposed by Heisenberg has no support from these experiments, and that the results of figure 5 suggest that the spectrum of the third derivative becomes flat for $k \geqslant 70$. There is little doubt that the correction for wire length applies also to Laufer's measurements (curve no. 7, figure 6).

As a subsidiary verification of the theory for wire-length effect, measurements $F(k)$ were made with two wires 1 and 6 mm long. The wave analyser was set on a particular frequency, and the two values of F were measured by simply switching from one wire to the other; this method minimizes the errors introduced by slow changes in the flow or the wire sensitivities. The same compensating network was used for both wires, since it was not possible to obtain an accurate adjustment of the compensating network in the presence of the large turbulence. The preamplifiers were operated with capacitive coupling (noise of 300 ohms), and therefore the measurements could not be extended beyond 15 kc/s. Figure 7 shows the measured values of η, where

$$\eta = \frac{[F(k)]_{l=6\,\mathrm{mm}}}{[F(k)]^{l=1\,\mathrm{mm}}}.$$

The theory of the wire-length effect predicts that η should start from 6 at low frequencies and begin to drop at $\omega = 250$ c/s, finally levelling off to $\sqrt{6}$ at high frequencies. The measurements support the theory.

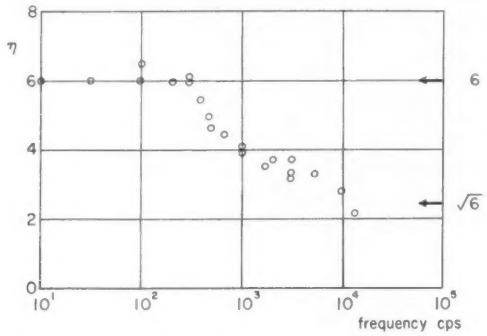

Figure 7. Effect of wire length on the measurement of a spectrum.

5. Skewness and kurtosis of $\partial u/\partial x$

Measurements of the mean-square of the square, cube and fourth powers of $\partial u/\partial x$ were made, using three chains of 10 double triodes, properly biased. Two chains were used for squaring and one for cubing, and the frequency responses were satisfactory. The largest errors were due to deviations from exact power law responses.

It was found that $S = -0{\cdot}45 \pm 0{\cdot}05$ and $\gamma = 4{\cdot}5 \pm 0{\cdot}5$; these results are not different from those obtained in grid flows with $30 < R_\lambda < 60$ or in shear flows (Betchov 1956).

The largest contribution to S originated from pulses of $\partial u/\partial t$ lasting approximately 0·5 millisecond. This corresponds to a length of about 2 mm and a correction for wire length may be necessary.

For the dissipation length parameter, the result was $\lambda = 0{\cdot}7$ cm which corresponds to a wave-number 1·5 cm^{-1} and to $R_\lambda = 250$. This result illustrates the fact that, at large values of R_λ, the length λ is intermediate between the energy scale k_E^{-1} (about 10 cm) and the scale of dissipation k_v^{-1} (about 0·025 cm). As is now generally appreciated, the name 'microscale' which is sometimes given to λ is not appropriate.

6. Molecular agitation in a perfect gas

In these experiments, the spectrum could not be measured beyond the frequency at which the noise of the hot-wire becomes dominant. It can be shown theoretically that a hollow wire having the same outside diameter as the wire used here and heated to the same temperature would give a better signal-to-noise ratio. Thus, by removing metal from within the wire, one could in theory extend the measurements of $F(k)$, with no obvious limit until the atomic structure of the wire becomes significant.

This raises the question of other forms of noise and in particular of the noise produced by the gas itself. Somewhere, the spectrum of turbulence must merge with the spectrum of molecular agitation, and we shall now attempt to locate this point.

Let us consider a volume of gas of linear dimension L containing n molecules. The molecular velocities are of the order of $(k'T/m)^{1/2}$, where k' is Boltzmann's constant, T the absolute temperature and m the mass of a molecule. The velocity of the gas can be defined as that of the centre-of-mass of the n molecules and it will fluctuate with a mean-square velocity of the order of $k'T/mn$. Consider now two such elements of volume, with centres distance D apart. The correlation between the velocities of the two centres of mass will vary from 1 if $D = 0$ to zero if $D > 2L$. From the correlation we can compute the three-dimensional spectrum for $kL \ll 1$; in this way we find

$$E_{\text{noise}} = (3k'T/4\pi^2\rho)k^2,$$

where $\rho = nm/L^3$ is the gas density. This spectrum increases with k^2 and should reach a maximum near $kL = 1$. For the present purpose, we can assume that L is small (say $L = 1$ micron, or 50 mean-free-paths in air).

If the one-dimensional spectrum of turbulence F falls as k^{-6}, the three dimensional spectrum (Batchelor 1953) is given by $E_{\text{turb.}} = 48F$. The spectrum of turbulence can therefore be expected to cross that of molecular agitation (so that $E_{\text{turb.}} = E_{\text{noise}}$) at the wave-number where

$$F(k) = (k'T/64\pi^2\rho)k^2. \tag{1}$$

This expression should be regarded as a crude estimate. It is based on a simplified picture of molecular agitation which does not include fluctuations of pressure, density or temperature. It is also based on the application of the Taylor hypothesis to the finest components of the turbulence. This certainly ceases to be valid of $k > U/\nu$, since such eddies are dissipated before they are displaced by the mean flow. In the present case this occurs at $k > 3000$ cm^{-1}.

The expression on the right-hand side of (1) is shown as dotted lines on figures 4 and 6 and it suggests that, unless $F(k)$ has an abrupt cut-off, the spectrum of turbulence disappears in molecular noise with a logarithmic slope of -6. This would leave quantities such as $\langle(\partial^3 u/\partial x^3)^2\rangle$ without significance in the sense of fluid mechanics. Stating it a different way, it appears that the gap between turbulence and molecular agitation may not be as wide as is generally assumed.

This work was jointly sponsored by the Navy Bureau of Ordnance under contract Nord 15872, and Project SQUID, which is jointly supported by the Office of Naval Research, the Office of Scientific Research (Air Force), and the Office of Ordnance Research (Army) under contract N6ori-105 TO III.

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Note on the position of ring singularities in an axisymmetric potential field

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SUMMARY

In this work a method is given for finding the position of ring singularities in a three-dimensional potential field having axial symmetry, by consideration of the very much easier case of a similar two-dimensional potential function. It is seen that the traces of these ring singularities on a plane through the axis of symmetry occur at points corresponding to those of the singularities existing in the two-dimensional plane when the axial velocity potential functions are the same. It is thought that this might be of value in the plotting of stream surfaces.

INTRODUCTION

When an arbitrary velocity potential function for perfect fluid flow is assumed, it is essential that the singularities of this function or its derivatives should be excluded from the fluid region that it is hoped may be reproduced physically. In the following work a method has been developed for obtaining the positions of the singularities in velocities, or velocity potential, by consideration of similar velocity or velocity potential distributions in the two-dimensional case. It will be supposed that the velocity potential function, or one of its derivatives, is known along the axis of symmetry, and it can be shown that this defines the velocity potential function throughout all space.

First it will be shown that a relationship exists between the two-dimensional case and the three-dimensional case with axial symmetry.

The procedure for then obtaining the ring singularities when $\phi = f(x)$ along the axis in three dimensions, with symmetry assumed, will be to consider the position of the singularities existing when $\phi = f(x)$ along the x-axis in two dimensions.

SIMILARITY OF CERTAIN TWO-DIMENSIONAL AND THREE-DIMENSIONAL VELOCITY POTENTIAL FUNCTIONS

The axis of symmetry will be taken as the x-axis in both the two- and three-dimensional cases. ϕ denotes a velocity potential function in both cases, and the value of ϕ along the axis of symmetry is supposed to be $2f(x)$.

If y is defined as the coordinate measured perpendicular to the x-axis in two dimensions and $z = x + iy$, it follows that $f(z) = \bar{f}(\bar{z})$, and so

$$\phi = f(z) + f(\bar{z}) \tag{1}$$

at any point of the two-dimensional field, with the usual notation for conjugate complex quantities. Here, $f(z)$ is the analytic function of z that takes the real value $f(x)$ when $y = 0$.

For the three-dimensional case with axial symmetry, the definition of the axial velocity potential as $2f(x)$ determines the potential function throughout the whole of space except along a number of lines. It may be shown (Whittaker & Watson 1927, p. 388) that

$$\phi = \frac{2}{\pi}\int_0^{\pi} f(x+ir\cos\theta)\,d\theta \tag{2}$$

at every point of the field at which the integral is analytic, where r is defined as the coordinate measured perpendicular to the x-axis in three dimensions. Putting $\lambda = \cos\theta$ in (2), one obtains

$$\phi = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x+ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda = \frac{2}{\pi}\left[\int_0^1 \frac{f(x+ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda + \int_0^1 \frac{f(x-ir\lambda)}{\sqrt{(1-\lambda^2)}}\,d\lambda\right]. \tag{3}$$

There is clearly great similarity between equations (1) and (3).

Suppose that $f(z)$ has a singularity at $z = \alpha + i\beta$; then $\lambda = 1$ gives the smallest value of r for which $\lambda r = \beta$, and when $\lambda = 1$, $f(x+ir\lambda)$ has a singularity at $x = \alpha$, $r = \beta$.

When no finite limit $f(\alpha+i\beta)$ exists, the integral (3) may not converge and then ϕ will be infinite at the singularity $x = \alpha$, $r = \beta$; note that the traces of this singularity are at the same points on a plane through the axis of symmetry as in the two-dimensional case. If however the integral (3) converges for the range $\lambda^2 \leqslant 1$ the velocity potential still exists. The condition that the integral (3) converges at $\lambda = 1$ is that

$$\lim_{\lambda\to 1} \frac{(1-\lambda)f(\alpha+i\beta)}{\sqrt{(1-\lambda^2)}}$$

is finite. Even in this case, however, it is easily seen that higher derivatives of ϕ must become infinite, so that the point is still a singularity.

Taking the velocity in the direction of the x-axis to be u $(= -\partial\phi/\partial x)$, the preceding work may be repeated with $f(x)$ replaced by $-f'(x)$. Hence the singularities in the velocity distributions in the field follow a similar pattern.

Some examples given below illustrate this.

Example 1

Suppose the axial velocity is given as $(1+x^2)^{-n} = -2f'(x)$, say.

Then, in two dimensions,

$$u = -\frac{\partial\phi}{\partial x} = \tfrac{1}{2}[(1+z^2)^{-n} + (1+\bar{z}^2)^{-n}].$$

This becomes infinite for $n > 0$ when $z^2 + 1 = 0$, or $y = \pm 1$, $x = 0$.

In three dimensions

$$f'(\alpha+i\beta\lambda) = -\tfrac{1}{2}[1+\alpha+i\beta\lambda)^2]^{-n}.$$

In this case

$$\lim_{\lambda\to 1} \frac{-(1-\lambda)^{1/2}}{2[1+(\alpha+i\beta\lambda)^2]^n} = \infty$$

for $n - \frac{1}{2} > 0$ when $\alpha = 0$, $\beta = 1$.

Hence a ring singularity exists symmetrically about the x-axis, of radius $r = 1$ in the plane $x = 0$, and the value of u is not finite at points on this ring provided $n - \frac{1}{2} > 0$. For the special case when $n = 1$, the integral (3) becomes

$$\frac{2}{\pi}\left[\int_0^1 \frac{d\lambda}{[(x+ir\lambda)^2+1]\sqrt{(1-\lambda^2)}} + \int_0^1 \frac{d\lambda}{[(x-ir\lambda)^2+1]\sqrt{(1-\lambda^2)}}\right],$$

and this can be found explicitly as

$$u(x,r) = \frac{1}{2i}\left[\{r^2+(v-i)^2\}^{-1/2} - \{r^2+(x+i)^2\}^{-1/2}\right],$$

which clearly has a singularity at $x = 0$, $r^2 = 1$. It is, in addition, discontinuous across the whole 'branch line' $x = 0$, $r > 1$.

Example 2

Suppose the axial velocity is given as $(1+e^x)^{-n} = -2f'(x)$, say.

Then, in two dimensions, $u = \frac{1}{2}\{(1+e^z)^{-n} + (1+e^{\bar{z}})^{-n}\}$; this becomes infinite for $n > 0$ when $e^z + 1 = 0$, or $x = 0$, $y = (2K+1)\pi$, where K is an integer.

For three dimensions, $f'(\alpha+i\beta\lambda) = -\frac{1}{2}(1+e^{\alpha+i\beta\lambda})^{-n}$ and

$$\lim_{\lambda\to 1} \frac{-1(-\lambda)^{1/2}}{2[1+e^{\alpha+i\beta\lambda}]^n} = \infty$$

for $n - \frac{1}{2} > 0$ when $\alpha = 0$ and $\beta = (2K+1)\pi$.

Hence ring singularities exist symmetrically about the x-axis, of radii $r = \pi, 3\pi, 5\pi, \ldots (2K+1)\pi$, in the plane $x = 0$.

A variation of this axial velocity has been used by Szczeniowski (1943) for designing wind tunnel contractions; but he does not appear to know where these singularities occur.

Example 3

Suppose the axial velocity is given as $2(1+x^4)^{-1} = -2f'(x)$, say.

In two dimensions, $u = (1+z^4)^{-1} + (1+\bar{z}^4)^{-1}$, which ceases to be analytic when $z^4 + 1 = 0$, that is, when $z = \exp\{\frac{1}{4}i(2K+1)\pi\}$, where K is an integer and $x + iy = (\pm 1 \pm i)/\sqrt{2}$.

For the three-dimensional case,

$$f'(\alpha+i\beta\lambda) = -\{1+(\alpha+i\beta\lambda)^4\}^{-1},$$

and for $\alpha = \pm 1/\sqrt{2}$, $\beta^2 = \frac{1}{2}$

$$\lim_{\lambda\to 1}\left\{\frac{-(1-\lambda)^{1/2}}{1+(\alpha+i\beta\lambda)^4}\right\} = \lim_{\lambda\to 1}\left\{\frac{-4(1-\lambda)^{1/2}}{4+(1\pm i\lambda)^4}\right\} = \infty.$$

Hence for this axial velocity there exist two ring singularities, given by $x = \pm 1/\sqrt{2}$, $r^2 = \frac{1}{2}$, at which the value of u is not bounded.

Conclusions

The procedure of defining an axial velocity distribution in three dimensions has been used by several authors in the design of nozzles and wind tunnel contractions. Prior to this work, it has been necessary to

calculate the stream functions (or potential functions) and then deduce the regions of the fluid in which the velocity components tend to infinity. Since these functions have often been in series form this has not always been easy and has usually been tedious. It is suggested here that, following the adoption of the axial velocity potential function, the position of the singularities in the two-dimensional (and hence of the three-dimensional) field should be found, as this will give a good indication of the regions where sharp rises in velocity along stream surfaces may occur. If, then, it is of importance to eliminate such rises, the plotting of a few stream surfaces in the region of the singularities alone might prevent a considerable wastage of time.

A point of interest is the reduction in degree of the singularities occurring in the three-dimensional case. For example, when the axial velocity is $(1+x^2)^{-2}$, then on the plane $x = 0$ the value of the $u(0, y)$ in two dimensions is $(1-y^2)^{-2}$ whilst it can be shown that the value of $u(0, r)$ in three dimensions is

$$\tfrac{1}{2}\{(1-r^2)^{-3/2}+(1-r^2)^{-1/2}\}.$$

The 'order of infinity' has therefore dropped from 2 to $\frac{3}{2}$; and a reduction in order of $\frac{1}{2}$ (compare example 1) seems general when going from three to two dimensions.

Whilst the preceding work was done mainly for the consideration of perfect fluid flow, it is quite general and could be applied to other problems with axial symmetry.

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REVIEWS

Combustion Processes, edited by B. LEWIS, R. N. PEASE and H. S. TAYLOR (Volume II of High Speed Aerodynamics and Jet Propulsion). Princeton University Press, 1956. 662 pp. $12.50 or 84*s*.

Almost every major advance in civilization has involved an increased understanding by man of the use of fire. The use of the camp fire for preparation of food was probably one of man's first steps away from animal ways. Fire was used as an aid in fashioning stone tools, and in the smelting of ores in the making of metal tools. The discovery of gunpowder made untenable the types of civilization existent at the time of its introduction. Although the industrial revolution was made possible by the steam engine which used fire only in a rather simple furnace, modern society has developed as the industrial furnace developed. The automobile and airplane are practical devices only because of the compact internal combustion engines.

It might be supposed that the scientific, quantitative knowledge of combustion processes would have reached an advanced stage in view of the great importance of fire. Much indeed is known, and great effort is being expended to increase this knowledge, but in spite of all effort the available information is in large part empirical.

The empirical nature of much of our knowledge and the amount of effort that has been put into the study of combustion processes is illustrated by the volume under review. This book would have been impossible 50 years ago. We have 662 pages devoted to a brief account of the state of knowledge of combustion processes in gases, liquids, and solids for premixed fuel and oxidizer and in non-premixed systems. And yet in spite of the great progress made since the last century, this book makes amply clear, in nearly every chapter, the unsatisfactory state of our present knowledge and frequently points the way to the obviously needed further research.

During the nineteenth century, while one group of physical chemists were struggling with the study of the kinetics of chemical reactions, another group were laying the foundations of thermodynamics and its application to chemical equilibrium. The present volume very appropriately opens with a discussion of the essential relations from thermodynamics that permit the calculation of the equilibrium composition of reaction products, the heat release and the flame temperature. Only the briefest treatment is given since the entire first volume of this series is entitled *Thermodynamics and Physics of Matter*.

In spite of the logical completeness of the understanding of chemical equilibrium afforded by thermodynamics, its practical usefulness is limited, as is made clear in the next two sections. From section C it is clear that, between the specification of a problem, involving a fuel and oxidizer composition, and the desired values of pressure, temperature, entropy and other properties of the combustion gases, lies a long path of computation. This path, even for simple systems for which all thermal and equilibrium constants are available, is so involved that it is seldom

pursued unless large scale automatic computers are available. The immediate future should see great progress in this area since the computing machines make practical many calculations which to date have been merely possible.

It would seem that section B on " Expansion Processes " was placed between A on " High Temperature Equilibrium " and C on " Computation Methods " so that the reader would see immediately the limitations as well as the importance of the thermodynamic methods. Section B, in a sense, is the real meat of jet propulsion ; it is the analysis of the production of the jet from high temperature combustion gases. Besides the usual compressible flow discussion of the flow of a gas through a nozzle, there is treated the appropriate modifications required by the fact that chemical equilibrium may or may not be preserved during the expansion. The additional problems presented by the fact that sometimes a condensed phase forms during the expansion are also considered.

The enormous progress that has been made during the past century in the understanding of the kinetic rate mechanism for various chemical processes is attested by the 60 pages devoted to the fundamentals of this subject. A short text on chemical kinetics would be a good name for these chapters. They start with an empirical introduction to reaction rate theory, introducing order, half life, temperature, rate constants, activation energy. This is followed by the theoretical understanding of the empirical facts through kinetic collision theory and absolute rate theory. By considering a series of specific cases, the ideas of free radicals, chain mechanisms, explosions, wall reactions, solid and gas reactions and experimental techniques are presented. Finally the bearing of all these effects on non-equilibrium in systems involving rapid changes is explained.

With the above as introduction, several reactions of importance in propulsion are studied in detail. The reader is brought up to date in our understanding of the hydrogen–oxygen, carbon monoxide–oxygen and hydrocarbon–oxygen systems. The uninitiated reader learns of the great complexity of even the simplest of these reactions and is thus brought to appreciate the present-day potentialities and limitations in the analysis of combustion systems. A brief treatment of several systems of interest in modern propulsion is included : boron hydrides, hydrogen–fluorine and nitromethane decomposition.

One more section on fundamental ideas is included, namely, " The Mechanics of Reacting Continua ". In this 11 page section the reader is taken through the ideas and derivations of the continuity, momentum and energy equations for flowing, reacting, viscous, heat conducting compressible fluids in general vector form. Only 7 pages are required for this, the last 4 pages of the chapter being devoted to a discussion of irreversible processes, including entropy transport, forces, currents, and the kinetic coefficients, and the introduction of the Onsager reciprocal relations. This section is an excellent very concise review of fundamentals from a general viewpoint for readers already well prepared in advanced theoretical fluid mechanics. As a text it's next to useless.

In present day combustion, a careful observer notes a discontinuity between the discussions of fundamentals and of applications. The fundamentals are rather precisely known physical principles, worked out in large part in precise mathematical form by carefully reasoned mathematical arguments. The applications on the other hand consist of many empirical facts ' explained ' by a series of *ad hoc* assumptions assisted by the fundamentals as by a crutch. This discontinuity was very sharp half a century ago. Today—as if by a diffusion process—the discontinuity has become dulled. The detailed nature of the blurred discontinuity in ideas and methods occupies the last two thirds of this volume.

Almost all of the middle third of the book is devoted to the combustion of gases. First the phenomena encountered in non-turbulent gases are discussed. A rather brief account of ' the flame ' is given which indicates the qualitative ideas of flame thickness, ignition temperature, etc. Only the simplest and least adequate relation for the flame speed, that of Mallard and LeChatelier, is derived, although a very adequate bibliography of the literature on more adequate flame speed relations is included. There follows a discussion of ideas and assumptions and corresponding derivations on a host of important topics; quenching distance, flame stabilization, blowoff and flashback, flame shape and fluid velocity field, propagation in channels, ignition, limits of inflammability, minimum ignition energy, etc. For those readers who approach combustion from the more exact field of aerodynamics, this chapter will leave them reeling. However, in most cases combustion is complicated, our understanding is not complete, and those who desire a more exact understanding should regard these phenomena as challenging and the *ad hoc* assumptions as the best treatment yet possible, a zeroth order approximation, so to speak. Some 20 pages devoted to quantitative experimental results on laminar flames serve to make these available for use and to show the need for more analytical studies.

The problems associated with the turbulent flame are, as would be expected, more complex and less well established than for laminar flames. In the turbulent case, even the simplest ideas, such as the flame ' speed ', are hard to define and measure. The fact that a conflict rages between the proponents of the wrinkled laminar flame sheet and the proponents of the extended reaction zone is not surprising. The author of this section (H) is a wrinkled flame sheet man, and most of the 50 pages of the section is devoted to an examination of available data in the light of the most modern form of this theory. It should be stated that so far the flame sheet wrinklers have succeeded in deriving quantitative consequences of their ideas in a reasonably basic way, while the extended reaction zone proponents have not. In the reviewer's opinion, the turbulent flame contains significant proportions of both wrinkled flame sheets and extended reaction zones. The success of one group in deriving formulas should not be interpreted as showing that turbulent flames do not involve extended reaction zones.

The diffusion flame, i.e. the combustion of a jet of fuel in air (say), is controlled more by the aerodynamic mixing processes than by the chemical rates. Hence there is a clear separation into laminar, transition

and turbulent diffusion flames. The experimental results are compared with present theoretical ideas of the parts played by molecular and turbulent diffusion. Again the understanding, as for example of flame height, is adequate to zero order but leaves much to be learned in the future.

Liquid fuels are usually burned by first atomizing and evaporating them. The study of such processes has yielded more precise knowledge than some other areas of combustion. However the added complexity still leaves much for future study. The same is true of solid fuels both in lumps and as dust. The treatment given both of these subjects is such as to bring the uninitiated reader a long way toward the understanding he will need for further work in these areas.

The application of the fundamental and empirical knowledge of combustion processes to jet propulsion is discussed in two long sections, one on liquid propellants and one on solid propellants. As is so often the case, the knowledge of fundamental processes serves more as a guide than as a quantitative calculation procedure on complete engineering systems. The complexity is of too high an order. Monopropellants, bipropellants and common solid propellants are considered. Equipment, ignition and ignition delay problems, chemical and physical mechanisms, pressure dependence, thermal decomposition, and performance characteristics are all treated. The performance attainable and the problems remaining are clearly presented.

The chemical part of this book on combustion closes with a qualitative discussion of detonation processes. Because of the small thickness of the detonation front, the overall gas dynamics of the detonation process is perhaps simpler than the more extended combustion zones of ordinary fires but the chemical and physical processes are at least as complex. Often the chemical and physical nature of explosives is complex and the processes confined to narrow zones are hard to study experimentally. The parts played by composition, reaction time, lateral expansion, detonation characteristics, including initiation and decay, are presented from a physical chemist's (rather than aerodynamicist's) point of view.

The last section gets into a book on combustion as a most modern touch, " Energy Production by Nuclear Reactions ". This section in a book on combustion processes in a series on high speed flight and jet propulsion suggests the inference that the future development of rocket vehicles both for terrestrial and space flight could be influenced in a major way by nuclear developments. There are some close relationships between combustion and nuclear energy production from both the hydrodynamic and transport mechanism points of view. The present chapter of 40 pages can, however, do no more than review the elementary physical principles, and this it does in a simple and clear manner.

This book is the best presentation of the whole of the knowledge of combustion currently available. The general excellence of the selection of material and its presentation will leave this book in first place until the growth of the field makes the present understanding inadequate.

H. W. Emmons

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